The Viking Battle - Part 1 2023 - Solutions

Problem 1 Let $k \geq 2$ be an interger. Find the smallest integer $n \geq k+1$ with the property that there exists a set of n distinct real numbers such that each of its elements can be written as a sum of k other distinct elements of the set.

Solution to problem 1

Answer: $n = k + 4$.

First we show that $n \geq k + 4$. Suppose that there exists such a set with n numbers $a_1 < a_2 < \cdots < a_n$.

Note that in order to express a_1 as a sum of k distinkt elements of the set, we must have $a_1 \geq a_2 + \cdots + a_{k+1}$ and, similarly for a_n, we must have $a_{n-k} + \cdots + a_{n-1} \geq a_n$.

If $n = k+1$ then $a_1 \geq a_2 + \cdots + a_n > a_1 + \cdots + a_{n-1} \geq a_n$, which is a contradiction.

If $n = k + 2$ then $a_1 \geq a_2 + \cdots + a_{n-1} \geq a_n$, which is also a contradiction.

If $n = k + 3$ then we have $a_1 \geq a_2 + \cdots + a_{n-2}$ and $a_3 + \cdots + a_{n-1} \geq a_n$, Adding the two inequalities we get $a_1 + a_{n-1} \geq a_2 + a_n$, again a contradiction.

It remains to give an example of a set with $k+4$ element satisfying the condition. We start with the even case $n = 2m$: Consider the set

$$S = \{-m, -(m-1), \ldots, -2, -1, 1, 2, \ldots, m-1, m\},$$

and remember that $m \geq 3$. If $s \in S \backslash \{-m, 1, 2\}$, then $s = (s-1) + 1$. We now add $2(m-3)$ elements from the set, such that their sum is zero, and this can be done since we have $m - 3$ untouched pairs $\pm l$ from the set. Similarly if $s \in \{-m, 1, 2\}$, then $s = (s+1) + (-1)$ and we add $2(m-3)$ elements from the set as before.

In the odd case $n = 2m + 1$, we just add 0 to the set S and to all sums, except when we want k numbers with sum 0. It is easy to see that the set contains k different numbers with sum 0. This shows that $n = k + 4$.

Problem 2 Let P be the set of all primes. Find all positive integers n such that $n!$ divides

$$\prod_{\substack{p<q\le n\\ p,q\in P}} (p+q).$$

Solution to problem 2

Answer: This only holds for $n = 7$.

Assume that $n!$ divides

$$\prod_{p<q\le n} (p+q),$$

and let $2 = p_1 < p_2 < \ldots < p_m \le n$ be the primes less than or equal to n. Now p_i divides $n!$ for all $i = 1, 2, \ldots, m$. In particular $p_m \mid p_i + p_j$ for some primes $p_i < p_j \le p_m$, and hence $p_m = p_i + p_j$, which implies $m \ge 3$, $p_i = 2$ and $p_m = 2 + p_{m-1}$.

Similarly $p_{m-1} \mid p_l + p_k$ for some $p_l < p_k \le m$, and

$$0 < \frac{p_k + p_l}{p_{m-1}} \le \frac{p_{m-1} + p_m}{p_{m-1}} = \frac{2p_{m-1} + 2}{p_{m-1}} < 3.$$

Hence $p_{m-1} = p_l + p_k$ or $2p_{m-1} = p_l + p_k$. As above, $p_{m-1} = p_l + p_k$ gives $p_{m-1} = 2 + p_{m-2}$. If $2p_{m-1} = p_l + p_k$, then $p_{m-1} < p_k$, so $p_k = p_m$ and

$$2p_{m-1} = p_l + p_m = p_l + p_{m-1} + 2 \ \Rightarrow p_{m-1} = p_l + 2 = p_{m-2} + 2.$$

Either way, $p_{m-2} > 2$ and 3 divides one of the primes p_{m-2}, p_{m-1}, p_m. Hence $p_{m-2} = 3$, $p_{m-1} = 5$ and $p_m = 7$. Now $7 \le n < 11$. Notice that

$$\prod_{p<q\le 7} (p+q) = (2+3)(2+5)(2+7)(3+5)(3+7)(5+7) = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7,$$

which is divisible by $7! = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, but not $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. Thus $n = 7$ is the only positive integer for which the condition holds.

Problem 3 Let n be a positive integer. We start with n piles of pebbles, each initially containing a single pebble. One can perform moves of the following form: Choose two piles, take an equal number of pebbles from each pile and form a new pile out of these pebbles. For each positive integer n, find the smallest number of non-empty piles that one can obtain by performing a finite sequence of moves of this form.

Solution to problem 3

Answer: If n is a power of 2, the answer is one pile. Otherwise it is two piles.

Since we can always combine two plies of 2^k pebbles each to a pile of 2^{k+1} pebbles, it is easy to see that for $n = 2^m$:

$$\begin{aligned}&2^m \text{ piles of 1 pebble} \to 2^{m-1} \text{ piles of 2 pebbles} \to \cdots \to\\&2 \text{ piles of } 2^{m-1} \text{ pebbles} \to 1 \text{ pile of } 2^m \text{ pebbles.}\end{aligned}$$

Assume that n is not a power of 2. First we prove that it is possible to make 2 piles. Choose N such that $2^N < n < 2^{N+1}$. Let $m = n - 2^N$. Then $0 < m < 2^N$ and we can make two piles like this:

$2^N + m$ piles of 1 pebble $\to$

1 pile of 2^N pebbles and m piles of 1 pebble $\to$

1 pile of $2^N - 1$ pebbles, $m - 1$ piles of 1 pebble and 1 pile of 2 pebbles $\to$

1 pile of $2^N - 2$ pebbles, m piles of 1 pebble and 1 pile of 2 pebbles $\to$

$\cdots \to$

1 pile of m pebbles, $2^N - 2$ piles of 1 pebble and 1 pile of 2 pebbles $\to$

1 pile of m pebbles and 2^{N-1} piles of 2 pebbles $\to$

1 pile of m pebbles and 1 pile of 2^N pebbles.

To finish the proof, we show that if n is not a power of 2, then it is not possible to make one pile. In one move we take c pebbles from two piles containing a and b pebbles, respectively:

$$a \to a - c, \quad b \to b - c, \quad 0 \to 2c.$$

If an odd number m divides the number of pebbles in each pile after the move, then m divides c and hence m also divides a and b. Hence if m is an odd number that divides the number of pebbles in each pile, then m also divided the number of pebbles in each pile in any previous position. If n is not a power of 2, it has an odd divisor $m > 1$. If it was possible to end with one pile of n pebbles, then m would divide the number of pebbles in any previous pile, and hence m would divide 1, a contradiction.