Problem 1 Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again in R. Prove that the lines CD, AQ and BR are concurrent.

Solution to problem 1

Common remarks. The introductory steps presented here are used in all solutions below.

Since $AC = BC = AD$, we have $\angle ABC = \angle BAC = \angle ACD = \angle ADC$. Since the quadrilaterals $APRQ$ and $AQCD$ are cyclic, we obtain

$$\angle CRA = 180^\circ - \angle ARP = 180^\circ - \angle AQP = \angle DQA = \angle DCA = \angle CBA,$$

so the points A, B, C, and R lie on some circle γ.

Solution 1. Introduce the point $X = AQ \cap CD$; we need to prove that B, R and X are collinear.

By means of the circle $(APRQ)$ we have

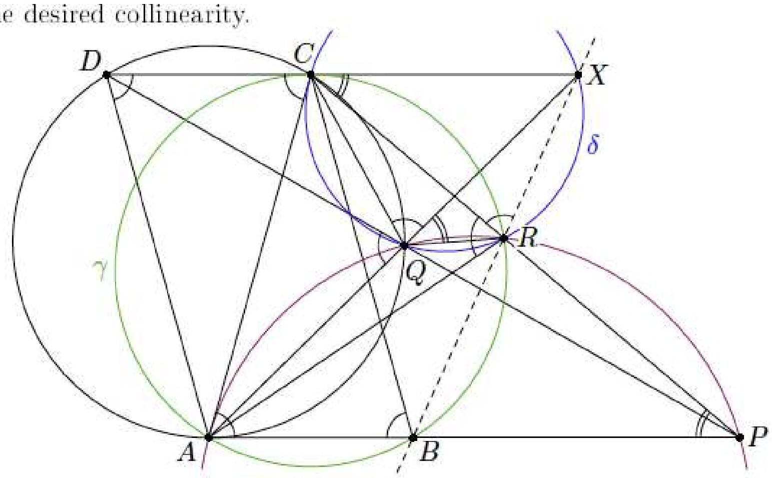
$$\angle RQX = 180^\circ - \angle AQR = \angle RPA = \angle RCX$$

(the last equality holds in view of $AB \parallel CD$), which means that the points C, Q, R, and X also lie on some circle δ.

Using the circles δ and γ we finally obtain

$$\angle XRC = \angle XQC = 180^\circ - \angle CQA = \angle ADC = \angle BAC = 180^\circ - \angle CRB,$$

that proves the desired collinearity.



Solution 2. Let α denote the circle $(APRQ)$. Since

$$\angle CAP = \angle ACD = \angle AQD = 180^\circ - \angle AQP,$$

the line AC is tangent to α.

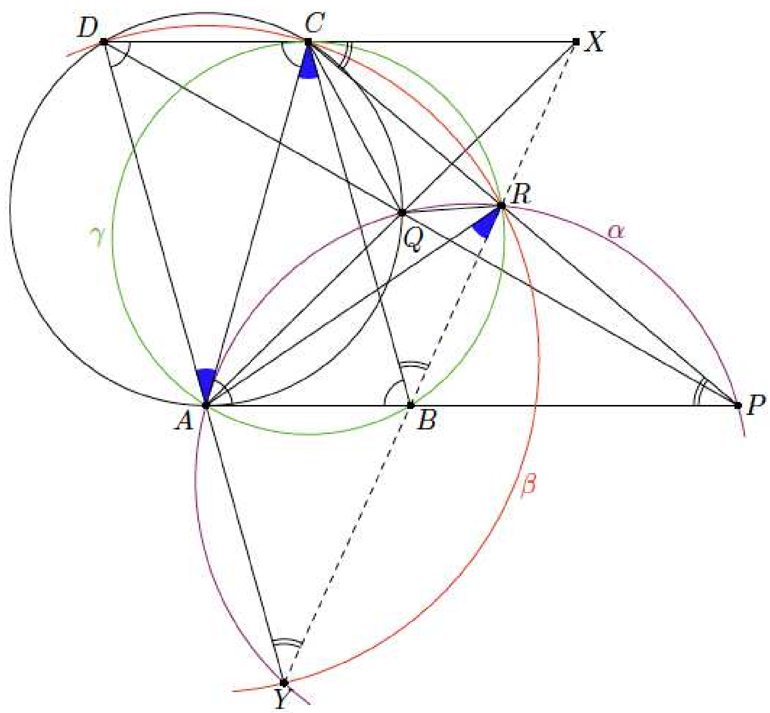
Now, let AD meet α again at a point Y (which necessarily lies on the extension of DA beyond A). Using the circle γ, along with the fact that AC is tangent to α, we have

$$\angle ARY = \angle CAD = \angle ACB = \angle ARB,$$

so the points Y, B, and R are collinear.

Applying Pascal's theorem to the hexagon $AAYRPQ$ (where AA is regarded as the tangent to α at A), we see that the points $AA \cap RP = C$, $AY \cap PQ = D$, and $YR \cap QA$ are collinear. Hence the lines CD, AQ, and BR are concurrent.

Comment 1. Solution 2 consists of two parts: (1) showing that BR and DA meet on α; and (2) showing that this yields the desired concurrency. Solution 3 also splits into those parts, but the proofs are different.



Solution 3. As in Solution 1, we introduce the point $X = AQ \cap CD$ and aim at proving that the points B, R, and X are collinear. As in Solution 2, we denote $\alpha = (APQR)$; but now we define Y to be the second meeting point of RB with α.

Using the circle α and noticing that CD is tangent to γ, we obtain

$$\angle RYA = \angle RPA = \angle RCX = \angle RBC. \tag{1}$$

So $AY \parallel BC$, and hence Y lies on DA.

Now the chain of equalities (1) shows also that $\angle RYD = \angle RCX$, which implies that the points C, D, Y, and R lie on some circle β. Hence, the lines CD, AQ, and YBR are the pairwise radical axes of the circles $(AQCD)$, α, and β, so those lines are concurrent.

Problem 2 Alice is given a rational number $r > 1$ and a line with two points $\mathcal{B} \neq \mathcal{R}$, where point $\mathcal{R}$ contains a red bead and point $\mathcal{B}$ contains a blue bead. Alice plays a solitary game by performing a sequence of moves. In every move, she chooses a (not necessarily) positive integer k, and a bead to move. If that bead is placed at point X, and the other bead is placed at point Y, then Alice moves the chosen bead to point X' with $\overrightarrow{YX'} = r^k\overrightarrow{YX}$.

Alice's goal is to move the red bead to the point $\mathcal{B}$. Find all rational numbers $r > 1$ such that Alice can reach her goal in at most 2021 moves.

Solution to problem 2

Answer: All $r = (b+1)/b$ with $b = 1, \ldots, 1010$.

Solution. Denote the red and blue beads by $\mathcal{R}$ and $\mathcal{B}$, respectively. Introduce coordinates on the line and identify the points with their coordinates so that $R = 0$ and $B = 1$. Then, during the game, the coordinate of $\mathcal{R}$ is always smaller than the coordinate of $\mathcal{B}$. Moreover, the distance between the beads always has the form r^ℓ with $\ell \in \mathbb{Z}$, since it only multiplies by numbers of this form. Denote the value of the distance after the m^{th} move by $d_m = r^{\alpha_m}$, $m = 0, 1, 2, \ldots$ (after the 0^{th} move we have just the initial position, so $\alpha_0 = 0$).

If some bead is moved in two consecutive moves, then Alice could instead perform a single move (and change the distance from d_i directly to d_{i+2}) which has the same effect as these two moves. So, if Alice can achieve her goal, then she may as well achieve it in fewer (or the same) number of moves by alternating the moves of $\mathcal{B}$ and $\mathcal{R}$. In the sequel, we assume that Alice alternates the moves, and that $\mathcal{R}$ is shifted altogether t times.

If $\mathcal{R}$ is shifted in the m^{th} move, then its coordinate increases by $d_m - d_{m+1}$. Therefore, the total increment of $\mathcal{R}$'s coordinate, which should be 1, equals

$$\text{either} \qquad (d_0 - d_1) + (d_2 - d_3) + \cdots + (d_{2t-2} - d_{2t-1}) = 1 + \sum_{i=1}^{t-1} r^{\alpha_{2i}} - \sum_{i=1}^{t} r^{\alpha_{2i-1}},$$

$$\text{or} \qquad (d_1 - d_2) + (d_3 - d_4) + \cdots + (d_{2t-1} - d_{2t}) = \sum_{i=1}^{t} r^{\alpha_{2i-1}} - \sum_{i=1}^{t} r^{\alpha_{2i}},$$

depending on whether $\mathcal{R}$ or $\mathcal{B}$ is shifted in the first move. Moreover, in the former case we should have $t \leqslant 1011$, while in the latter one we need $t \leqslant 1010$. So both cases reduce to an equation

$$\sum_{i=1}^{n} r^{\beta_i} = \sum_{i=1}^{n-1} r^{\gamma_i}, \qquad \beta_i, \gamma_i \in \mathbb{Z}, \tag{1}$$

for some $n \leqslant 1011$. Thus, if Alice can reach her goal, then this equation has a solution for $n = 1011$ (we can add equal terms to both sums in order to increase n).

Conversely, if (1) has a solution for $n = 1011$, then Alice can compose a corresponding sequence of distances $d_0, d_1, d_2, \ldots, d_{2021}$ and then realise it by a sequence of moves. So the problem reduces to the solvability of (1) for $n = 1011$.

Assume that, for some rational r, there is a solution of (1). Write r in lowest terms as $r = a/b$. Substitute this into (1), multiply by the common denominator, and collect all terms on the left hand side to get

$$\sum_{i=1}^{2n-1} (-1)^i a^{\mu_i} b^{N-\mu_i} = 0, \qquad \mu_i \in \{0, 1, \ldots, N\}, \tag{2}$$

for some $N \geqslant 0$. We assume that there exist indices j_- and j_+ such that $\mu_{j_-} = 0$ and $\mu_{j_+} = N$.

Reducing (2) modulo $a-b$ (so that $a \equiv b$), we get

$$0 = \sum_{i=1}^{2n-1} (-1)^i a^{\mu_i} b^{N-\mu_i} \equiv \sum_{i=1}^{2n-1} (-1)^i b^{\mu_i} b^{N-\mu_i} = -b^N \mod (a-b).$$

Since $\gcd(a-b, b) = 1$, this is possible only if $a - b = 1$.

Reducing (2) modulo $a+b$ (so that $a \equiv -b$), we get

$$0 = \sum_{i=1}^{2n-1} (-1)^i a^{\mu_i} b^{N-\mu_i} \equiv \sum_{i=1}^{2n-1} (-1)^i (-1)^{\mu_i} b^{\mu_i} b^{N-\mu_i} = S b^N \mod (a+b)$$

for some odd (thus nonzero) S with $|S| \leqslant 2n-1$. Since $\gcd(a+b, b) = 1$, this is possible only if $a+b \mid S$. So $a+b \leqslant 2n-1$, and hence $b = a-1 \leqslant n-1 = 1010$.

Thus we have shown that any sought r has the form indicated in the answer. It remains to show that for any $b = 1, 2, \ldots, 1010$ and $a = b+1$, Alice can reach the goal. For this purpose, in (1) we put $n = a$, $\beta_1 = \beta_2 = \cdots = \beta_a = 0$, and $\gamma_1 = \gamma_2 = \cdots = \gamma_b = 1$.

Comment 1. Instead of reducing modulo $a+b$, one can reduce modulo a and modulo b. The first reduction shows that the number of terms in (2) with $\mu_i = 0$ is divisible by a, while the second shows that the number of terms with $\mu_i = N$ is divisible by b.

Notice that, in fact, $N > 0$, as otherwise (2) contains an alternating sum of an odd number of equal terms, which is nonzero. Therefore, all terms listed above have different indices, and there are at least $a+b$ of them.

Comment 2. Another way to investigate the solutions of equation (1) is to consider the Laurent polynomial

$$L(x) = \sum_{i=1}^{n} x^{\beta_i} - \sum_{i=1}^{n-1} x^{\gamma_i}.$$

We can pick a sufficiently large integer d so that $P(x) = x^d L(x)$ is a polynomial in $\mathbb{Z}[x]$. Then

$$P(1) = 1, \tag{3}$$

and

$$1 \leqslant |P(-1)| \leqslant 2021. \tag{4}$$

If $r = p/q$ with integers $p > q \geqslant 1$ is a rational number with the properties listed in the problem statement, then $P(p/q) = L(p/q) = 0$. As $P(x)$ has integer coefficients,

$$(p - qx) \mid P(x). \tag{5}$$

Plugging $x = 1$ into (5) gives $(p-q) \mid P(1) = 1$, which implies $p = q+1$. Moreover, plugging $x = -1$ into (5) gives $(p+q) \mid P(-1)$, which, along with (4), implies $p + q \leqslant 2021$ and $q \leqslant 1010$. Hence $x = (q+1)/q$ for some integer q with $1 \leqslant q \leqslant 1010$.

Problem 3 A hunter and a rabbit play a game on an infinite grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share a side). The hunter wins if after some finite time either

- the rabbit cannot move; or
- the hunter can determine the cell in which the rabbit started.

Decide whether there exits a winning strategy for the hunter.

Solution to problem 3

Answer: Yes, there exists a colouring that yields a winning strategy for the hunter.

Solution. A central idea is that several colourings $C_1, C_2, \ldots, C_k$ can be merged together into a single product colouring $C_1 \times C_2 \times \cdots \times C_k$ as follows: the colours in the product colouring are ordered tuples $(c_1, \ldots, c_n)$ of colours, where c_i is a colour used in C_i, so that each cell gets a tuple consisting of its colours in the individual colourings C_i. This way, any information which can be determined from one of the individual colourings can also be determined from the product colouring.

Now let the hunter merge the following colourings:

- The first two colourings C_1 and C_2 allow the tracking of the horizontal and vertical movements of the rabbit.

  The colouring C_1 colours the cells according to the residue of their x-coordinates modulo 3, which allows to determine whether the rabbit moves left, moves right, or moves vertically. Similarly, the colouring C_2 uses the residues of the y-coordinates modulo 3, which allows to determine whether the rabbit moves up, moves down, or moves horizontally.

- Under the condition that the rabbit's x-coordinate is unbounded, colouring C_3 allows to determine the exact value of the x-coordinate:

  In C_3, the columns are coloured white and black so that the gaps between neighboring black columns are pairwise distinct. As the rabbit's x-coordinate is unbounded, it will eventually visit two black cells in distinct columns. With the help of colouring C_1 the hunter can catch that moment, and determine the difference of x-coordinates of those two black cells, hence deducing the precise column.

  Symmetrically, under the condition that the rabbit's y-coordinate is unbounded, there is a colouring C_4 that allows the hunter to determine the exact value of the y-coordinate.

- Finally, under the condition that the sum $x + y$ of the rabbit's coordinates is unbounded, colouring C_5 allows to determine the exact value of this sum: The diagonal lines $x + y = \mathrm{const}$ are coloured black and white, so that the gaps between neighboring black diagonals are pairwise distinct.