NORDIC MATHEMATICAL CONTEST

PROBLEMS AND SOLUTIONS, 1987–2011

PROBLEMS

The problems are identified as $xy.n.$, whery x and y are the last digits of the competition year and n is the n:th problem of that year.

NMC 1, March 30, 1987

87.1. Nine journalists from different countries attend a press conference. None of these speaks more than three languages, and each pair of the journalists share a common language. Show that there are at least five journalists sharing a common language.

87.2. Let $ABCD$ be a parallelogram in the plane. We draw two circles of radius R, one through the points A and B, the other through B and C. Let E be the other point of intersection of the circles. We assume that E is not a vertex of the parallelogram. Show that the circle passing through A, D, and E also has radius R.

87.3. Let f be a strictly increasing function defined in the set of natural numbers satisfying the conditions $f(2) = a > 2$ and $f(mn) = f(m)f(n)$ for all natural numbers m and n. Determine the smallest possible value of a.

87.4. Let a, b, and c be positive real numbers. Prove:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \le \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

NMC 2, April 4, 1988

88.1. The positive integer n has the following property: if the three last digits of n are removed, the number $\sqrt[3]{n}$ remains. Find n.

88.2. Let a, b, and c be non-zero real numbers and let $a \ge b \ge c$. Prove the inequality

$$\frac{a^3 - c^3}{3} \ge abc\left(\frac{a-b}{c} + \frac{b-c}{a}\right).$$

When does equality hold?

88.3. Two concentric spheres have radii r and R, $r < R$. We try to select points A, B and C on the surface of the larger sphere such that all sides of the triangle ABC would be tangent to the surface of the smaller sphere. Show that the points can be selected if and only if $R \le 2r$.

88.4. Let m_n be the smallest value of the function

$$f_n(x) = \sum_{k=0}^{2n} x^k.$$

Show that $m_n \to \frac{1}{2}$, as $n \to \infty$.

NMC 3, April 10, 1989

89.1. Find a polynomial P of lowest possible degree such that
(a) P has integer coefficients,
(b) all roots of P are integers,
(c) $P(0) = -1$,
(d) $P(3) = 128$.

89.2. Three sides of a tetrahedron are right-angled triangles having the right angle at their common vertex. The areas of these sides are A, B, and C. Find the total surface area of the tetrahedron.

89.3. Let S be the set of all points t in the closed interval $[-1, 1]$ such that for the sequence $x_0, x_1, x_2, \ldots$ defined by the equations $x_0 = t$, $x_{n+1} = 2x_n^2 - 1$, there exists a positive integer N such that $x_n = 1$ for all $n \geq N$. Show that the set S has infinitely many elements.

89.4. For which positive integers n is the following statement true: if $a_1, a_2, \ldots, a_n$ are positive integers, $a_k \leq n$ for all k and $\sum_{k=1}^{n} a_k = 2n$, then it is always possible to choose $a_{i_1}, a_{i_2}, \ldots, a_{i_j}$ in such a way that the indices $i_1, i_2, \ldots, i_j$ are different numbers, and $\sum_{k=1}^{j} a_{i_k} = n$?

NMC 4, April 5, 1990

90.1. Let m, n, and p be odd positive integers. Prove that the number

$$\sum_{k=1}^{(n-1)^p} k^m$$

is divisible by n.

90.2. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove

$$\sqrt[3]{a_1^3 + a_2^3 + \ldots + a_n^3} \leq \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}. \tag{1}$$

When does equality hold in (1)?

90.3. Let ABC be a triangle and let P be an interior point of ABC. We assume that a line l, which passes through P, but not through A, intersects AB and AC (or their extensions over B or C) at Q and R, respectively. Find l such that the perimeter of the triangle AQR is as small as possible.

90.4. It is possible to perform three operations f, g, and h for positive integers: $f(n) = 10n$, $g(n) = 10n + 4$, and $h(2n) = n$; in other words, one may write 0 or 4 in the end of the number and one may divide an even number by 2. Prove: every positive integer can be constructed starting from 4 and performing a finite number of the operations f, g, and h in some order.

NMC 5, April 10, 1991

91.1. Determine the last two digits of the number

$$2^5 + 2^{5^2} + 2^{5^3} + \cdots + 2^{5^{1991}},$$

written in decimal notation.

91.2. In the trapezium $ABCD$ the sides AB and CD are parallel, and E is a fixed point on the side AB. Determine the point F on the side CD so that the area of the intersection of the triangles ABF and CDE is as large as possible.

91.3. Show that

$$\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < \frac{2}{3}$$

for all $n \geq 2$.

91.4. Let $f(x)$ be a polynomial with integer coefficients. We assume that there exists a positive integer k and k consecutive integers n, $n + 1$, $\ldots$, $n + k - 1$ so that none of the numbers $f(n)$, $f(n + 1)$, $\ldots$, $f(n + k - 1)$ is divisible by k. Show that the zeroes of $f(x)$ are not integers.

NMC 6, April 8, 1992

92.1. Determine all real numbers $x > 1$, $y > 1$, and $z > 1$, satisfying the equation

$$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} = 2\left(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}\right).$$

92.2. Let $n > 1$ be an integer and let a_1, a_2, $\ldots$, a_n be n different integers. Show that the polynomial

$$f(x) = (x - a_1)(x - a_2) \cdot \ldots \cdot (x - a_n) - 1$$

is not divisible by any polynomial with integer coefficients and of degree greater than zero but less than n and such that the highest power of x has coefficient 1.

92.3. Prove that among all triangles with inradius 1, the equilateral one has the smallest *perimeter*.

92.4. Peter has many squares of equal side. Some of the squares are black, some are white. Peter wants to assemble a big square, with side equal to n sides of the small squares, so that the big square has *no* rectangle formed by the small squares such that all the squares in the vertices of the rectangle are of equal colour. How big a square is Peter able to assemble?

NMC 7, March 17, 1993

93.1. Let F be an increasing real function defined for all x, $0 \le x \le 1$, satisfying the conditions

(i) $F\left(\frac{x}{3}\right) = \frac{F(x)}{2}$,

(ii) $F(1-x) = 1 - F(x)$.

Determine $F\left(\frac{173}{1993}\right)$ and $F\left(\frac{1}{13}\right)$.

93.2. A hexagon is inscribed in a circle of radius r. Two of the sides of the hexagon have length 1, two have length 2 and two have length 3. Show that r satisfies the equation

$$2r^3 - 7r - 3 = 0.$$

93.3. Find all solutions of the system of equations

$$\begin{cases} s(x) + s(y) = x \\ x + y + s(z) = z \\ s(x) + s(y) + s(z) = y - 4, \end{cases}$$

where x, y, and z are positive integers, and $s(x)$, $s(y)$, and $s(z)$ are the *numbers of digits* in the decimal representations of x, y, and z, respectively.

93.4. Denote by $T(n)$ the *sum of the digits of the decimal representation* of a positive integer n.

a) Find an integer N, for which $T(k \cdot N)$ is even for all k, $1 \le k \le 1992$, but $T(1993 \cdot N)$ is odd.

b) Show that no positive integer N exists such that $T(k \cdot N)$ is even for all positive integers k.

NMC 8, March 17, 1994

94.1. Let O be an interior point in the equilateral triangle ABC, of side length a. The lines AO, BO, and CO intersect the sides of the triangle in the points A_1, B_1, and C_1. Show that

$$|OA_1| + |OB_1| + |OC_1| < a.$$

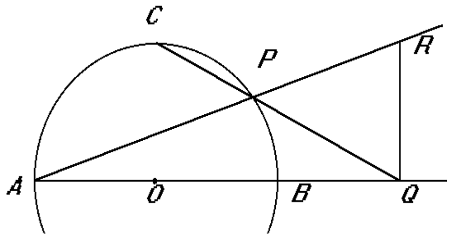
94.2. We call a finite plane set S consisting of points with integer coefficients a *two-neighbour set*, if for each point (p, q) of S exactly two of the points $(p+1, q)$, $(p, q+1)$, $(p-1, q)$, $(p, q-1)$ belong to S. For which integers n there exists a two-neighbour set which contains exactly n points?

94.3. A piece of paper is the square $ABCD$. We fold it by placing the vertex D on the point D' of the side BC. We assume that AD moves on the segment $A'D'$ and that $A'D'$ intersects AB at E. Prove that the perimeter of the triangle EBD' is one half of the perimeter of the square.

94.4. Determine all positive integers $n < 200$, such that $n^2 + (n+1)^2$ is the square of an integer.

NMC 9, March 15, 1995

95.1. Let AB be a diameter of a circle with centre O. We choose a point C on the circumference of the circle such that OC and AB are perpendicular to each other. Let P be an arbitrary point on the (smaller) arc BC and let the lines CP and AB meet at Q. We choose R on AP so that RQ and AB are perpendicular to each other. Show that $|BQ| = |QR|$.



95.2. Messages are coded using sequences consisting of zeroes and ones only. Only sequences with at most two consecutive ones or zeroes are allowed. (For instance the sequence 011001 is allowed, but 011101 is not.) Determine the number of sequences consisting of exactly 12 numbers.

95.3. Let $n \geq 2$ and let $x_1, x_2, \ldots x_n$ be real numbers satisfying $x_1 + x_2 + \ldots + x_n \geq 0$ and $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Let $M = \max\{x_1, x_2, \ldots, x_n\}$. Show that

$$M \geq \frac{1}{\sqrt{n(n-1)}}. \tag{1}$$

When does equality hold in (1)?

95.4. Show that there exist infinitely many mutually non-congruent triangles T, satisfying

(i) The side lengths of T are consecutive integers.

(ii) The area of T is an integer.

NMC 10, April 11, 1996

96.1. Show that there exists an integer divisible by 1996 such that the sum of the its decimal digits is 1996.

96.2. Determine all real numbers x, such that

$$x^n + x^{-n}$$

is an integer for all integers n.

96.3. The circle whose diameter is the altitude dropped from the vertex A of the triangle ABC intersects the sides AB and AC at D and E, respectively ($A \neq D$, $A \neq E$). Show that the circumcentre of ABC lies on the altitude dropped from the vertex A of the triangle ADE, or on its extension.

96.4. The real-valued function f is defined for positive integers, and the positive integer a satisfies

$$f(a) = f(1995), \qquad f(a+1) = f(1996), \qquad f(a+2) = f(1997)$$

$$f(n+a) = \frac{f(n)-1}{f(n)+1} \quad \text{for all positive integers } n.$$

(i) Show that $f(n+4a) = f(n)$ for all positive integers n.

(ii) Determine the smallest possible a.

NMC 11, April 9, 1997

97.1. Let A be a set of seven positive numbers. Determine the maximal number of triples (x, y, z) of elements of A satisfying $x < y$ and $x + y = z$.

97.2. Let $ABCD$ be a convex quadrilateral. We assume that there exists a point P inside the quadrilateral such that the areas of the triangles ABP, BCP, CDP, and DAP are equal. Show that at least one of the diagonals of the quadrilateral bisects the other diagonal.

97.3. Let A, B, C, and D be four different points in the plane. Three of the line segments AB, AC, AD, BC, BD, and CD have length a. The other three have length b, where $b > a$. Determine all possible values of the quotient $\frac{b}{a}$.

97.4. Let f be a function defined in the set $\{0, 1, 2, \ldots\}$ of non-negative integers, satisfying $f(2x) = 2f(x)$, $f(4x+1) = 4f(x)+3$, and $f(4x-1) = 2f(2x-1)-1$. Show that f is an injection, i.e. if $f(x) = f(y)$, then $x = y$.

NMC 12, April 2, 1998

98.1. Determine all functions f defined in the set of rational numbers and taking their values in the same set such that the equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ holds for all rational numbers x and y.

98.2. Let C_1 and C_2 be two circles intersecting at A and B. Let S and T be the centres of C_1 and C_2, respectively. Let P be a point on the segment AB such that $|AP| \neq |BP|$ and $P \neq A$, $P \neq B$. We draw a line perpendicular to SP through P and denote by C and D the points at which this line intersects C_1. We likewise draw a line perpendicular to TP through P and denote by E and F the points at which this line intersects C_2. Show that C, D, E, and F are the vertices of a rectangle.

98.3. (a) For which positive numbers n does there exist a sequence x_1, x_2, $\ldots$, x_n, which contains each of the numbers 1, 2, $\ldots$, n exactly once and for which $x_1 + x_2 + \cdots + x_k$ is divisible by k for each $k = 1, 2, \ldots, n$?

(b) Does there exist an infinite sequence x_1, x_2, x_3, $\ldots$, which contains every positive integer exactly once and such that $x_1 + x_2 + \cdots + x_k$ is divisible by k for every positive integer k?

98.4. Let n be a positive integer. Count the number of numbers $k \in \{0, 1, 2, \ldots, n\}$ such that $\binom{n}{k}$ is odd. Show that this number is a power of two, i.e. of the form 2^p for some nonnegative integer p.

NMC 13, April 15, 1999

99.1. The function f is defined for non-negative integers and satisfies the condition

$$f(n) = \begin{cases} f(f(n+11)), & \text{if } n \leq 1999 \\ n-5, & \text{if } n > 1999. \end{cases}$$

Find all solutions of the equation $f(n) = 1999$.

99.2. Consider 7-gons inscribed in a circle such that all sides of the 7-gon are of different length. Determine the maximal number of 120° angles in this kind of a 7-gon.

99.3. The infinite integer plane $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ consists of all number pairs (x, y), where x and y are integers. Let a and b be non-negative integers. We call any move from a point (x, y) to any of the points $(x \pm a, y \pm b)$ or $(x \pm b, y \pm a)$ a (a, b)-knight move. Determine all numbers a and b, for which it is possible to reach all points of the integer plane from an arbitrary starting point using only (a, b)-knight moves.

99.4. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $n \geq 1$. Show that

$$n\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \geq \left(\frac{1}{1+a_1} + \cdots + \frac{1}{1+a_n}\right)\left(n + \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right).$$

When does equality hold?

NMC 14, March 30, 2000

00.1. In how many ways can the number 2000 be written as a sum of three positive, not necessarily different integers? (Sums like $1+2+3$ and $3+1+2$ etc. are the same.)

00.2. The persons $P_1, P_1, \ldots, P_{n-1}, P_n$ sit around a table, in this order, and each one of them has a number of coins. In the start, P_1 has one coin more than P_2, P_2 has one coin more than P_3, etc., up to P_{n-1} who has one coin more than P_n. Now P_1 gives one coin to P_2, who in turn gives two coins to P_3 etc., up to P_n who gives n coins to P_1. Now the process continues in the same way: P_1 gives $n+1$ coins to P_2, P_2 gives $n+2$ coins to P_3; in this way the transactions go on until someone has not enough coins, i.e. a person no more can give away one coin more than he just received. At the moment when the process comes to an end in this manner, it turns out that there are two neighbours at the table such that one of them has exactly five times as many coins as the other. Determine the number of persons and the number of coins circulating around the table.

00.3. In the triangle ABC, the bisector of angle B meets AC at D and the bisector of angle C meets AB at E. The bisectors meet each other at O. Furthermore, $OD = OE$. Prove that either ABC is isosceles or $\angle BAC = 60^\circ$.

00.4. The real-valued function f is defined for $0 \leq x \leq 1$, $f(0) = 0$, $f(1) = 1$, and

$$\frac{1}{2} \leq \frac{f(z) - f(y)}{f(y) - f(x)} \leq 2$$

for all $0 \leq x < y < z \leq 1$ with $z - y = y - x$. Prove that

$$\frac{1}{7} \leq f\left(\frac{1}{3}\right) \leq \frac{4}{7}.$$

NMC 15, March 29, 2001

01.1. Let A be a finite collection of squares in the coordinate plane such that the vertices of all squares that belong to A are (m, n), $(m+1, n)$, $(m, n+1)$, and $(m+1, n+1)$ for some integers m and n. Show that there exists a subcollection B of A such that B contains at least 25 % of the squares in A, but no two of the squares in B have a common vertex.

01.2. Let f be a bounded real function defined for all real numbers and satisfying for all real numbers x the condition

$$f\left(x+\frac{1}{3}\right)+f\left(x+\frac{1}{2}\right)=f(x)+f\left(x+\frac{5}{6}\right).$$

Show that f is periodic. (A function f is bounded, if there exists a number L such that $|f(x)| < L$ for all real numbers x. A function f is periodic, if there exists a positive number k such that $f(x+k) = f(x)$ for all real numbers x.)

01.3. Determine the number of real roots of the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} = 0.$$

01.4. Let $ABCDEF$ be a convex hexagon, in which each of the diagonals AD, BE, and CF divides the hexagon into two quadrilaterals of equal area. Show that AD, BE, and CF are concurrent.

NMC 16, April 4, 2002

02.1. The trapezium $ABCD$, where AB and CD are parallel and $AD < CD$, is inscribed in the circle c. Let DP be a chord of the circle, parallel to AC. Assume that the tangent to c at D meets the line AB at E and that PB and DC meet at Q. Show that $EQ = AC$.

02.2. In two bowls there are in total N balls, numbered from 1 to N. One ball is moved from one of the bowls into the other. The average of the numbers in the bowls is increased in both of the bowls by the same amount, x. Determine the largest possible value of x.

02.3. Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ be real numbers, and let $a_1, a_2, \ldots, a_n$ be all different. Show that if all the products

$$(a_i + b_1)(a_i + b_2)\cdots(a_i + b_n),$$

$i = 1, 2, \ldots, n$, are equal, then the products

$$(a_1 + b_j)(a_2 + b_j)\cdots(a_n + b_j),$$

$j = 1, 2, \ldots, n$, are equal, too.

02.4. Eva, Per and Anna play with their pocket calculators. They choose different integers and check, whether or not they are divisible by 11. They only look at nine-digit numbers consisting of all the digits 1, 2, ..., 9. Anna claims that the probability of such a number to be a multiple of 11 is exactly 1/11. Eva has a different opinion: she thinks the probability is less than 1/11. Per thinks the probability is more than 1/11. Who is correct?

NMC 17, April 3, 2003

03.1. Stones are placed on the squares of a chessboard having 10 rows and 14 columns. There is an odd number of stones on each row and each column. The squares are coloured black and white in the usual fashion. Show that the number of stones on black squares is even. Note that there can be more than one stone on a square.

03.2. Find all triples of integers (x, y, z) satisfying

$$x^3 + y^3 + z^3 - 3xyz = 2003.$$

03.3. The point D inside the equilateral triangle $\triangle ABC$ satisfies $\angle ADC = 150°$. Prove that a triangle with side lengths $|AD|$, $|BD|$, $|CD|$ is necessarily a right-angled triangle.

03.4. Let $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ be the set of non-zero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ satisfying

$$f(x) + f(y) = f(xy\,f(x+y)),$$

for $x, y \in \mathbb{R}^*$ and $x + y \neq 0$.

NMC 18, April 1, 2004

04.1. 27 balls, labelled by numbers from 1 to 27, are in a red, blue or yellow bowl. Find the possible numbers of balls in the red bowl, if the averages of the labels in the red, blue, and yellow bowl are 15, 3 ja 18, respectively.

04.2. Let $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$, for $n = 1, 2, \ldots$, be the Fibonacci sequence. Show that there exists a strictly increasing infinite arithmetic sequence none of whose numbers belongs to the Fibonacci sequence. [A sequence is *arithmetic*, if the difference of any of its consecutive terms is a constant.]

04.3. Let $x_{11}, x_{21}, \ldots, x_{n1}$, $n > 2$, be a sequence of integers. We assume that all of the numbers x_{i1} are not equal. Assuming that the numbers $x_{1k}, x_{2k}, \ldots, x_{nk}$ have been defined, we set

$$x_{i,k+1} = \frac{1}{2}(x_{ik} + x_{i+1,k}),\ i = 1, 2, \ldots, n-1,$$
$$x_{n,k+1} = \frac{1}{2}(x_{nk} + x_{1k}).$$

Show that for n odd, x_{jk} is not an integer for some j, k. Does the same conclusion hold for n even?

04.4. Let a, b, and c be the side lengths of a triangle and let R be its circumradius. Show that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}.$$

NMC 19. April 5, 2005

05.1. Find all positive integers k such that the product of the digits of k, in the decimal system, equals

$$\frac{25}{8}k - 211.$$

05.2. Let a, b, and c be positive real numbers. Prove that

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq a+b+c.$$

05.3. There are 2005 young people sitting around a (large!) round table. Of these at most 668 are boys. We say that a girl G is in a strong position, if, counting from G to either direction at any length, the number of girls is always strictly larger than the number of boys. (G herself is included in the count.) Prove that in any arrangement, there always is a girl in a strong position.

05.4. The circle $\mathcal{C}_1$ is inside the circle $\mathcal{C}_2$, and the circles touch each other at A. A line through A intersects $\mathcal{C}_1$ also at B and $\mathcal{C}_2$ also at C. The tangent to $\mathcal{C}_1$ at B intersects $\mathcal{C}_2$ at D and E. The tangents of $\mathcal{C}_1$ passing through C touch $\mathcal{C}_1$ at F and G. Prove that D, E, F, and G are concyclic.

NMC 20. March 30, 2006

06.1. Let B and C be points on two fixed rays emanating from a point A such that $AB + AC$ is constant. Prove that there exists a point $D \neq A$ such that the circumcircles of the triangels ABC pass through D for every choice of B and C.

06.2. The real numbers x, y and z are not all equal and they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k.$$

Determine all possible values of k.

06.3. A sequence of positive integers $\{a_n\}$ is given by

$$a_0 = m \qquad \text{and} \qquad a_{n+1} = a_n^5 + 487$$

for all $n \geq 0$. Determine all values of m for which the sequence contains as many square numbers as possible.

06.4. The squares of a 100×100 chessboard are painted with 100 different colours. Each square has only one colour and every colour is used exactly 100 times. Show that there exists a row or a column on the chessboard in which at least 10 colours are used.

NMC 21. March 29, 2007

07.1. Find one solution in positive integers to the equation

$$x^2 - 2x - 2007y^2 = 0.$$

07.2. A triangle, a line and three rectangles, with one side parallel to the given line, are given in such a way that the rectangles completely cover the sides of the triangle. Prove that the rectangles must completely cover the interior of the triangle.

07.3. The number 10^{2007} is written on a blackboard, Anne and Berit play a game where the player in turn makes one of two operations:

(i) replace a number x on the blackboard by two integer numbers a and b greater than 1 such that $x = ab$;

(ii) erase one or both of two equal numbers on the blackboard.

The player who is not able to make her turn loses the game. Who has a winning strategy?

07.4. A line through a point A intersects a circle in two points, B and C, in such a way that B lies between A and C. From the point A draw the two tangents to the circle, meeting the circle at points S and T. Let P be the intersection of the lines ST and AC. Show that $AP/PC = 2 \cdot AB/BC$.

NMC 22. March 31, 2008

08.1. Determine all real numbers A, B and C such that there exists a real function f that satisfies

$$f(x + f(y)) = Ax + By + C.$$

for all real x and y.

08.2. Assume that $n \geq 3$ people with different names sit around a round table. We call any unordered pair of them, say M and N, *dominating*, if

(i) M and N do not sit on adjacent seats, and

(ii) on one (or both) of the arcs connecting M and N along the table edge, all people have names that come alphabetically after the names of M and N.

Determine the minimal number of dominating pairs.

08.3. Let ABC be a triangle and let D and E be points on BC and CA, respectively, such that AD and BE are angle bisectors of ABC. Let F and G be points on the circumcircle of ABC such that AF and DE are parallel and FG and BC are parallel. Show that

$$\frac{AG}{BG} = \frac{AC + BC}{AB + CB}.$$

08.4. The difference between the cubes of two consecutive positive integers is a square n^2, where n is a positive integer. Show that n is the sum of two squares.

NMC 23. April l2, 2009

09.1. A point P is chosen in an arbitrary triangle. Three lines are drawn through P which are parallel to the sides of the triangle. The lines divide the triangle into three smaller triangles and three parallelograms. Let f be the ratio between the total area of the three smaller triangles and the area of the given triangle. Show that $f \geq \frac{1}{3}$ and determine those points P for which $f = \frac{1}{3}$.

09.2. On a faded piece of paper it is possible, with some effort, to discern the following:

$$(x^2 + x + a)(x^{15} - \ldots) \;=\; x^{17} + x^{13} + x^5 - 90x^4 + x - 90.$$

Some parts have got lost, partly the constant term of the first factor of the left side, partly the main part of the other factor. It would be possible to restore the polynomial forming the other factor, but we restrict ourselves to asking the question: What is the value of the constant term a? We assume that all polynomials in the statement above have only integer coefficients.

09.3. The integers 1, 2, 3, 4 and 5 are written on a blackboard. It is allowed to wipe out two integers a and b and replace them with $a + b$ and ab. Is it possible, by repeating this procedure, to reach a situation where three of the five integers on the blackboard are 2009?

09.4. There are 32 competitors in a tournament. No two of them are equal in playing strength, and in a one against one match the better one always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.

NMC 24. April 13, 2010

10.1. A function $f : \mathbb{Z} \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of positive integers, is non-decreasing and satisfies $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n. Prove that $f(8)f(13) \geq (f(10))^2$.

10.2. Three circles Γ_A, Γ_B and Γ_C share a common point of intersection O. The other common of Γ_A and Γ_B is C, that of Γ_A and Γ_C is B and that of Γ_C and Γ_B is A. The line AO intersects the circle Γ_C in the poin $X \neq O$. Similarly, the line BO intersects the circle Γ_B in the point $Y \neq O$, and the line CO intersects the circle Γ_C in the point $Z \neq O$. Show that

$$\frac{|AY||BZ||CX|}{|AZ||BX||CY|} = 1.$$

10.3. Laura has 2010 lamps connected with 2010 buttons in front of her. For each button, she wants to know the corresponding lamp. In order to do this, she observes which lamps are lit when Richard presses a selection of buttons. (Not pressing anything is also a possible selection.) Richard always presses the buttons simultaneously, so the lamps are lit simultaneously, too.

a) If Richard chooses the buttons to be pressed, what is the maximum number of different combinations of buttons he can press until Laura can assign the buttons to the lamps correctly?

b) Supposing that Laura will choose the combinations of buttons to be pressed, what is the minimum number of attempts she has to do until she is able to associate the buttons with the lamps in a correct way?

10.4. A positive integer is called *simple* if its ordinary decimal representation consists entirely of zeroes and ones. Find the least positive integer k such that each positive integer n can be written as $n = a_1 \pm a_2 \pm a_3 \pm \cdots \pm a_k$, where $a_1, \ldots, a_k$ are simple.

NMC 25. April 4, 2011

11.1. When $a_0, a_1, \ldots, a_{1000}$ denote digits, can the sum of the 1001-digit numbers $a_0a_1 \ldots a_{1000}$ and $a_{1000}a_{999} \ldots a_0$ have odd digits only?

11.2. In a triangle ABC assume $AB = AC$, and let D and E be points on the extension of segment BA beyond A and on the segment BC, respectively, such that the lines CD and AE are parallel. Prove that $CD \geq \frac{4h}{BC} CE$, where h is the height from A in triangle ABC. When does equality hold?

11.3. Find all functions f such that

$$f(f(x) + y) = f(x^2 - y) + 4yf(x)$$

for all real numbers x and y.

11.4. Show that for any integer $n \geq 2$ the sum of the fractions $\frac{1}{ab}$, where a and b are relatively prime positive integers such that $a < b \leq n$ and $a + b > n$, equals $\frac{1}{2}$.