Version: English

39th Nordic Mathematical Contest

25 March, 2025

Time allowed is 4 hours. Each problem is worth 7 points. The only permitted aids are writing and drawing materials.

1. Let n be a positive integer greater than 2. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$\Big(f(x+y)\Big)^n = f(x^n) + f(y^n), \quad \text{for all integers } x, y.$$

2. Let p be a prime and suppose $2^{2p} \equiv 1 \pmod{2p+1}$. Prove that $2p+1$ is prime. [1]

3. Let ABC be an acute triangle with orthocenter H and circumcenter O. Let E and F be points on the line segments AC and AB respectively such that $AEHF$ is a parallelogram. Prove that $|OE| = |OF|$.

4. Denote by S_n the set of all permutations of the set $\{1, 2, \ldots, n\}$. Let $\sigma \in S_n$ be a permutation. We define the *displacement* of σ to be the number $d(\sigma) = \sum_{i=1}^{n} |\sigma(i) - i|$. We say that σ is *maximally displacing* if $d(\sigma)$ is the largest possible, i.e. if $d(\sigma) \geq d(\pi)$, for all $\pi \in S_n$.

a) Suppose σ is a maximally displacing permutation of $\{1, 2, \ldots, 2024\}$. Prove that $\sigma(i) \neq i$, for all $i \in \{1, 2, \ldots, 2024\}$.

b) Does the statement of part a) hold for permutations of $\{1, 2, \ldots, 2025\}$?

[1] This is a special case of Pocklington's theorem. A proof of this special case is required.