The 35th Nordic Mathematical Contest

Friday, 16 April 2021

Time allowed: 4 hours. Each problem is worth 7 points.
Only writing and drawing tools are allowed.

Problem 1. On a blackboard a finite number of integers greater than one are written. Every minute, Nordi additionally writes on the blackboard the smallest positive integer greater than every other integer on the blackboard and not divisible by any of the numbers on the blackboard. Show that from some point onwards Nordi only writes primes on the blackboard.

Problem 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying that for every $x \in \mathbb{R}$,

$$f(x(1+|x|)) \le x \le f(x)(1+|f(x)|).$$

Problem 3. Let n be a positive integer. Alice and Bob play the following game. First, Alice picks $n+1$ subsets $A_1, \ldots, A_{n+1}$ of $\{1, \ldots, 2^n\}$ each of size 2^{n-1}. Second, Bob picks $n+1$ arbitrary integers $a_1, \ldots, a_{n+1}$. Finally, Alice picks an integer t. Bob wins if there exists an integer $1 \le i \le n+1$ and $s \in A_i$ such that $s + a_i \equiv t \pmod{2^n}$. Otherwise, Alice wins.
Find all values of n where Alice has a winning strategy.

Problem 4. Let A, B, C and D be points on the circle ω such that $ABCD$ is a convex quadrilateral. Suppose that AB and CD intersect at a point E such that that A is between B and E and that BD and AC intersect at a point F. Let $X \neq D$ be the point on ω such that DX and EF are parallel. Let Y be the reflection of D through EF and suppose that Y is inside the circle ω.
Show that A, X, and Y are collinear.