The 33rd Nordic Mathematical Contest

Monday, April 1st, 2019

English version

Time allowed: 4 hours. Each problem is worth 7 points.
Only writing and drawing tools are allowed.

Problem 1 A set of different positive integers is called *meaningful* if for any finite non-empty subset the corresponding arithmetic and geometric means are both integers.

a) Does there exist a meaningful set which consists of 2019 numbers?

b) Does there exist an infinite meaningful set?

Note: The geometric mean of the non-negative numbers $a_1, a_2, \ldots, a_n$ is defined as $\sqrt[n]{a_1 a_2 \cdots a_n}$

Problem 2 Let a, b, c be the side lengths of a right angled triangle with $c > a, b$.
Show that
$$3 < \frac{c^3 - a^3 - b^3}{c(c-a)(c-b)} \le \sqrt{2} + 2.$$

Problem 3 The quadrilateral $ABCD$ satisfies $\angle ACD = 2\angle CAB$, $\angle ACB = 2\angle CAD$ and $CB = CD$.
Show that $\angle CAB = \angle CAD$.

Problem 4 Let n be an integer with $n \ge 3$ and assume that $2n$ vertices of a regular $(4n + 1)$-gon are coloured. Show that there must exist three of the coloured vertices forming an isosceles triangle.