The 33rd Nordic Mathematical Contest

Monday, April 1st, 2019

English version

Time allowed: 4 hours. Each problem is worth 7 points.
Only writing and drawing tools are allowed.

Problem 1 A set of different positive integers is called *meaningful* if for any finite non-empty subset the corresponding arithmetic and geometric means are both integers.

a) Does there exist a meaningful set which consists of 2019 numbers?

b) Does there exist an infinite meaningful set?

Note: The geometric mean of the non-negative numbers $a_1, a_2, \ldots, a_n$ *is defined as* $\sqrt[n]{a_1 a_2 \cdots a_n}$

Problem 2 Let a, b, c be the side lengths of a right angled triangle with $c > a, b$.
Show that

$$3 < \frac{c^3 - a^3 - b^3}{c(c-a)(c-b)} \leq \sqrt{2} + 2.$$

Problem 3 The quadrilateral $ABCD$ satisfies $\angle ACD = 2\angle CAB$, $\angle ACB = 2\angle CAD$ and $CB = CD$.
Show that $\angle CAB = \angle CAD$.

Problem 4 Let n be an integer with $n \geq 3$ and assume that $2n$ vertices of a regular $(4n+1)$-gon are coloured. Show that there must exist three of the coloured vertices forming an isosceles triangle.

Solution 1 a) Notice that $\{2019! \cdot 1^{2019!}, 2019! \cdot 2^{2019!}, \dots, 2019! \cdot 2019^{2019!}\}$ is such a set. Observe that if all the elements are divisible by 2019! then the arithmetic means will be integer for all the subsets. Also, if A is a set such that the geometric means are integer for all non-empty subsets and the set B is obtained from the set A by multiplying each element with with a given integer c then all the non-empty subsets of B will have an integer geometric mean, since

$$\sqrt[k]{ca_{i_1}ca_{i_2}\cdots ca_{i_k}} = c\sqrt[k]{a_{i_1}a_{i_2}\cdots a_{i_k}}.$$

It is thus sufficient to find a set of 2019 positive integers such that the geometric mean of every non-empty subset in an integer. Now, for an integer a the number $\sqrt[k]{a^{2019!}} = a^{\frac{2019!}{k}}$ for all integers $1 \leq k \leq 2019$ so $\{1^{2019!}, 2^{2019!}, \dots, 2019^{2019!}\}$ is a set such that the geometric mean of every non-empty subset is an integer.
b) Assume there exist such a set A and let $n, m, a_1, a_2, \dots, a_{m-1}$ be distinct elements in A with $n < m$. Then $\frac{n+a_1+a_2+\cdots+a_{m-1}}{m}$ and $\frac{m+a_1+a_2+\cdots+a_{m-1}}{m}$ are integers and also their difference

$$\frac{m+a_1+a_2+\cdots+a_{m-1}}{m} - \frac{n+a_1+a_2+\cdots+a_{m-1}}{m} = \frac{m-n}{m}.$$

Therefore, we have $\frac{n}{m}$ is an integer and since m and n are positive integers we have $m \leq n$ which is a contradiction.

Solution 2 Observe that

$$c^3 - a^3 - b^3 = ca^2 + cb^2 - a^3 - b^3 = a^2(c-a) + b^2(c-b) =$$

$$(c^2 - b^2)(c-a) + (c^2 - b^2)(c-b) = (c-a)(c-b)(c+a+c+b)$$

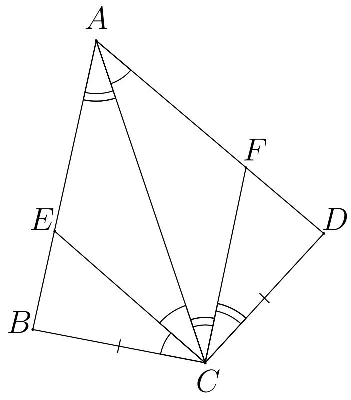
Therefore, it is equivalent to prove that

$$c < a + b \leq \sqrt{2}c$$

The left inequality we get from the triangle inequality and the right inequality we get from

$$a + b \leq \sqrt{2}c \Leftrightarrow (a+b)^2 \leq 2c^2 = 2a^2 + 2b^2 \Leftrightarrow 0 \leq (a-b)^2$$

Solution 3 Let the angle bisectors from angle C in triangle ACB and ACD intersect AB and AD in points E and F respectively. From $\angle ACE = \angle CAD$ it follows that CE and AD are parallel. Similarly CF and AB are parallel. Hence $AECF$ is a parallelogram. From this it follows that $\angle BEC = \angle BAD = \angle CFD$.



The angle bisector theorem yields

$$\frac{BE}{CF} = \frac{BE}{AE} = \frac{CB}{CA} = \frac{CD}{CA} = \frac{DF}{AF} = \frac{DF}{CE},$$

which gives

$$|BE| \cdot |CE| = |DF| \cdot |CF|.$$

By the sine area formula we obtain that BCE have DCF have equal area. Hence triangles BCA and DCA also have equal areal. By the sine area formula we now get

$$\sin(\angle ACB) = \sin(\angle DCA)$$

Since $ABCD$ is a quadrilateral, $\angle ACB + \angle DCA \neq 180$ and hence we conclude from the above that $\angle CAB = \angle CAD$.

Solution 4 Assume that it is possible to color $2n$ of the vertices of a $4n+1$-gon such that there are no three colored vertices forming an isosceles triangle. Enumerate the vertices consecutively as $H_{-2n}, H_{-2n+1}, \ldots, H_0, H_1, H_2, \ldots, H_{2n}$ and consider first the case where there are two colored neighboring vertices. Assume wlog that the vertices H_0 and H_1 are colored. Then at most one of the vertices H_{-i} and H_i are colored for all $i = 1, 2, \ldots, 2n$ since these form an isosceles triangle with H_0. Similarly at most one of H_{-i} and H_{i+2} are colored for all $i = 1, 2, \ldots, 2n-2$, and at most one of $H_{-(2n-1)}$ and H_{-2n} are colored since these form an isosceles triangles with H_1. The three vertices H_0, H_1, H_i, $i = 2, -1, -2n$ also form isoceles triangles and hence H_{-1}, H_2, H_{-2n} are not colored. It follows that no consecutive vertices in the two strings

$$H_{-2} - H_4 - H_{-4} - H_6 - \ldots - H_{2n-2} - H_{-(2n-2)} - H_{2n}$$
$$H_3 - H_{-3} - H_5 - H_{-5} - \ldots - H_{2n-1} - H_{-(2n-1)}$$

are colored. Since each string contains an even amount of vertices, at most half of each string are colored and this is obtained only when every second vertex is colored in each string. By counting we see that each string contains $2n-2$ vertices and we conclude that every second vertex is colored in each string. Because $n \geq 3$ at least one of the isoceles triangles $H_0H_{-2}H_{-4}$, $H_1H_3H_5$ or $H_{2n-2}H_{2n}H_{-(2n-1)}$ must be colored. Hence there are no colored neighboring vertices.

If there are no colored neighboring vertices we can assume wlog that H_i is colored for all odd i, but then $H_1H_3H_5$ is an isosceles colored triangle. Hence we have a contradiction, showing that there must neccesarily exists 3 colored vertices forming an isosceles triangle.