Solutions to the

2004 Nordic Mathematical Contest

Problem 1

Let r be the number of balls in the red bowl, b be the number of balls in the blue bowl and y be the number of balls in the yellow bowl. Because the mean of the 5 smallest integers is 3 we have $b \leq 5$. We have

$$\begin{aligned} r + b + y &= 27 \\ 15r + 3b + 18y &= 27 \cdot 14 \end{aligned}$$

From this we get

$$\begin{aligned} 4r + 5y &= 99 \\ b &= 27 - r - y \\ b &\leq 5 \end{aligned}$$

The only positive solutions are $(r, b, y) = (11, 5, 11), (16, 4, 7), (21, 3, 3)$
The 3 values of r are all possible.

$r = 11:$	Blue : $\{1, 2, 3, 4, 5\}$	Red : $\{10, 11, \cdots, 18, 19, 20\}$
$r = 16:$	Blue : $\{1, 2, 4, 5\}$	Red : $\{7, 8, \cdots, 14, 16, 17, \cdots, 23\}$
$r = 21:$	Blue : $\{2, 3, 4\}$	Red : $\{5, 6, \cdots, 25\}$

Problem 2

A sequence $\{a_k\}$ is arithmetic if $a_{k+1} - a_k = d$ for all k, where d is some constant, so $a_k = dk + a_0$. Notice that the arithmetic sequence is constant modulo its fixed increase, $a_k \equiv a_0 \pmod{d}$ for all k. So to find an increasing arithmetic sequence with no term in common with the Fibonacci sequence, it suffices to find integers $d > 0$ and a_0 such that f_n is never equivalent to a_0 modulo d.

Calculate the Fibonacci sequence modulo 8:

$$0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, \ldots$$

We have again reached: $0, 1, \ldots$ and because of the relation $f_{n+2} = f_{n+1} + f_n$ the sequence now repeats itself modulo 8. Notice that 4 does not appear, so the arithmetic sequence $a_k = 8k + 4$ has no term in common with the Fibonacci sequence.

Problem 3

Let $M_k = \max_j x_{jk}$ and $m_k = \min_j x_{jk}$. It is clear that M_k is a non-increasing and m_k a non-decreasing sequence. Also, $M_{k+1} = M_k$ only if $x_{jk} = x_{j+1,k} = M_k$ for some j. If exactly p "adjacent" x_{ik}'s equal M_k, then only $p-1$ adjacent $x_{i,k+1}$'s equal M_k. Eventually we reach a step, where $M_{k+1} < M_k$. Similarly, $m_{k+1} > m_k$ at some stage. Now if all the numbers in all the sequences are integers, so must be the maxima and minima. After a finite number of steps the maximum and minimum are equal, and so are all the numbers. We then have for some k

$$x_{1k} + x_{2k} = x_{2k} + x_{3k} = \cdots = x_{nk} + x_{1k}.$$

If n is odd, this gives $x_{1k} = x_{3k} = \cdots = x_{nk} = x_{2k} = \cdots = x_{n-1,k}$. Working backwards, the numbers in the starting sequence have to be equal.

But if n is even, then the sequence $0, 2, 0, 2, \ldots, 0, 2$ is a counterexample because in next step all the numbers will be equal to 1 and we never get a number that is not an integer.

Remark: It can be shown that the only counterexamples are sequences of the type $a, b, a, b, \ldots, a, b$ with $a \equiv b \pmod 2$:

In the argument above, if n is even, we get

$$x_{1k} = x_{3k} = \cdots = x_{n-1,k} = a \quad \text{and} \quad x_{2k} = x_{4k} = \cdots = x_{nk} = b.$$

But if $k > 1$, then $x_{1k} = \frac{1}{2}(x_{1,k-1} + x_{2,k-1})$, $x_{2k} = \frac{1}{2}(x_{2,k-1} + x_{3,k-1})$ etc., and

$$\frac{n}{2}a = x_{1k} + x_{3k} + \cdots + x_{n-1,k} = \frac{1}{2}(x_{1,k-1} + x_{2,k-1} + \cdots + x_{n,k-1}),$$

$$\frac{n}{2}b = x_{2k} + x_{4k} + \cdots + x_{nk} = \frac{1}{2}(x_{2,k-1} + x_{3,k-1} + \cdots + x_{n,k-1} + x_{1,k-1},$$

so $a = b$. Therefore it is only possible that $a \neq b$ when $k = 1$ and clearly to get integers in step $k+1$ we must have $a \equiv b \pmod 2$.

Alternate solution for the odd case: Assume that $x_{1k}, \ldots, x_{nk}$ are integers for some $k > 1$. Then $x_{i,k-1} \equiv x_{i+1,k-1} \pmod 2$ for all i, and if $k > 2$

$$x_{i,k-2} + x_{i+1,k-2} \equiv x_{i+1,k-2} + x_{i+2,k-2} \pmod 4 \qquad \text{for all } i.$$

Since n is odd, it follows that $x_{i,k-2} \equiv x_{i+1,k-2} \pmod 4$ for all i. By induction it follows that $x_{i,k-j} \equiv x_{i+1,k-j} \pmod{2^j}$ for all i and all $j < k$. Hence if $x_{i,k}$ is an integer for all i and k, then $x_{i,1} = x_{i+1,1} \pmod{2^j}$ for all i and j, thus they must all be equal.

Problem 4

Let A, B and C be the vertices of the triangle and call the angles at the vertices α, β and γ, respectively. Let O be the centre of the circumcircle, so $|OA| = |OB| = |OC| = R$. Draw the perpendiculars from O to each of the sides. This gives us three pairs of right angled triangles, from which we get that $a = 2R\sin\alpha$, $b = 2R\sin\beta$ and $c = 2R\sin\gamma$. Using this the inequality can be transformed to

$$\sin\,\alpha + \sin\,\beta + \sin\,\gamma \geq 4\sin\,\alpha\sin\,\beta\sin\,\gamma.$$

Using the GM-AM inequality we get

$$4\sin\,\alpha\sin\,\beta\sin\,\gamma \leq 4\left(\frac{\sin\,\alpha + \sin\,\beta + \sin\,\gamma}{3}\right)^3.$$

We will show that

$$\frac{\sin\,\alpha + \sin\,\beta + \sin\,\gamma}{3} \leq \frac{\sqrt{3}}{2},$$

which is equivalent to

$$4\left(\frac{\sin\,\alpha + \sin\,\beta + \sin\,\gamma}{3}\right)^3 \leq \sin\,\alpha + \sin\,\beta + \sin\,\gamma,$$

which then will give the wanted inequality.

We find that

$$\begin{aligned} \sin\,\alpha + \sin\,\beta + \sin\,\gamma &= 2\sin\,\frac{\alpha+\beta}{2}\cos\,\frac{\alpha-\beta}{2} + 2\sin\,\frac{\alpha+\beta}{2}\cos\,\frac{\alpha+\beta}{2} \\ &\leq 2\sin\,\frac{\alpha+\beta}{2}\left(1 + \cos\,\frac{\alpha+\beta}{2}\right). \end{aligned}$$

The function $f(t) = 2\sin\,t(1+\cos\,t) = 2\sin\,t + \sin\,2t$ has the derivative $f'(t) = 2\cos\,t + 2\cos\,2t$. The equation $f'(t) = 0$ has the unique solution $t_0 = \frac{\pi}{3}$ in the interval $(0, \pi)$; comparing $f(0) = 0$, $f(\pi) = 0$ and $f(\frac{\pi}{3}) = \frac{3\sqrt{3}}{2}$ gives that f has its largest value in the interval for $t_0 = \frac{\alpha+\beta}{2} = \frac{\pi}{3}$, with equality for $\alpha = \beta = \frac{\pi}{3}$.

Remark: Instead of using derivatives it is possible to prove

$$\sin\,\alpha + \sin\,\beta + \sin\,\gamma \leq \frac{3\sqrt{3}}{2}$$

using Jensen's inequality because $\sin(x)$ is concave for $0° \leq x \leq 180°$:

$$\frac{\sin\,\alpha + \sin\,\beta + \sin\,\gamma}{3} \leq \sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = \sin(60°) = \frac{\sqrt{3}}{2}$$

Alternate solution: Using $F = sr = \frac{1}{2}(a+b+c)r$ and $F = \frac{abc}{4R}$ we get

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{2s}{abc} = \frac{1}{2Rr}$$

so the problem can be solved using the fact that $2r \leq R$. But this follows from Euler's formula $|OI|^2 = R(R-2r) \geq 0$, where O is the circumcentre and I is the incentre