

The Georg Mohr Contest 2024
Second Round

Tuesday, 9 January 2024 at 9–13

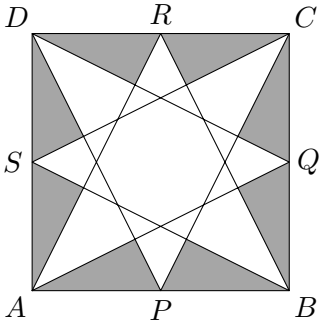
Aids permitted: only writing and drawing tools.
Remember that your arguments are important in the assessment
and that points may also be awarded to partial answers.

Problem 1. A *miracle number* is a number formed from a positive integer as follows: First one adds 3 to the number, then the result is multiplied by 4, then 5 is added to the new result, and finally one multiplies by 6. A *wonder number* is a number formed from a positive integer as follows: First the number is multiplied by 3, then 4 is added to the result, then the new result is multiplied by 5, and finally one adds 6.

a) Prove that the number 318 is a miracle number, but not a wonder number.

b) Do there exist numbers which are both miracle numbers and wonder numbers?

Problem 2. The square $ABCD$ has side length 20. The points P, Q, R and S are the midpoints of the sides. What is the area of the white star?



Problem 3. Georg writes a 100 letter word where he uses 60 A's, 12 B's and 28 C's. The word is a palindrome, i.e. a word which reads the same backwards as forwards. If he removes the last 34 letters, the word is still a palindrome. Is it possible to determine which letter is the 34th in the word?

Problem 4. Three different numbers are written on a blackboard. Each of the numbers are either 0 or a positive integer. In each move, Alma may choose two numbers a and b, where $a > b$, and replace them by $a^2 - b^2$ and $3a - b$. After a finite number of moves Alma achieves that there are three identical numbers on the blackboard. Which numbers may have been on the blackboard at the beginning?

Problem 5. Let $0 < x_1 < x_2 < \cdots < x_{2024} = 1$ be real numbers. A *colouring* of these numbers consists of painting 1012 of the numbers red and the remaining 1012 numbers blue. The red numbers are denoted in ascending order $r_1 < \cdots < r_{1012}$, and the blue numbers in ascending order $b_1 < \cdots < b_{1012}$. Two examples of colourings are:

$$\underbrace{x_1, \ldots, x_{1012}}_{r_1,\ldots,r_{1012}}, \underbrace{x_{1013}, \ldots, x_{2024}}_{b_1,\ldots,b_{1012}} \quad \text{and} \quad \underbrace{x_1}_{r_1}, \underbrace{x_2}_{b_1}, \underbrace{x_3}_{r_2}, \underbrace{x_4}_{b_2}, \ldots, \underbrace{x_{2023}}_{r_{1012}}, \underbrace{x_{2024}}_{b_{1012}}.$$

For any colouring one considers the quantity

$$S = r_1 b_1 + r_2(b_2 - b_1) + r_3(b_3 - b_2) + \cdots + r_{1012}(b_{1012} - b_{1011}).$$

a) Prove that one may choose the numbers $x_1, \ldots, x_{2024}$ such that any colouring satisfies $S < \frac{1}{2}$.

b) Prove that no matter how the numbers $x_1, \ldots, x_{2024}$ are chosen, there exists a colouring so that $S > \frac{x_{2023}}{2}$.

Sponsors: Undervisningsministeriet, Novo Nordisk Fonden, LEO Fondet, Jobindex, Institut for Matematiske Fag KU, Institut for Matematik og Datalogi SDU, Institut for Matematik AU, Institut for Matematiske Fag AAU, Georg Mohr Fonden, the family of a former member of Georg Mohr-arbejdsgruppen.