"Baltic Way – 92" Mathematical Team Contest

Vilnius, November 5–8, 1992

1. Let p, q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least 3 natural numbers > 1 (not necessarily different).

2. Denote by $d(n)$ the number of all positive divisors of a natural number n (including 1 and n). Prove that there are infinitely many n, such that $n/d(n)$ is an integer.

3. Find an infinite non-constant arithmetic progression of natural numbers such that each term is neither a sum of two squares, nor a sum of two cubes (of natural numbers).

4. Is it possible to draw a hexagon with vertices in the knots of an integer lattice so that the squares of the lengths of the sides are six consecutive positive integers?

5. It is given that $a^2+b^2+(a+b)^2 = c^2+d^2+(c+d)^2$. Prove that $a^4+b^4+(a+b)^4 = c^4+d^4+(c+d)^4$.

6. Prove that the product of the 99 numbers $\dfrac{k^3-1}{k^3+1}$, $k = 2, 3, \ldots, 100$ is greater than $2/3$.

7. Let $a = \sqrt[1992]{1992}$. Which number is greater

$$a^{a^{a^{\cdots^{a}}}}\Big\}_{1992} \quad \text{or} \quad 1992\ ?$$

8. Find all integers satisfying the equation $2^x \cdot (4-x) = 2x+4$.

9. A polynomial $f(x) = x^3+ax^2+bx+c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial f has three different real roots.

10. Find all fourth degree polynomials $p(x)$ such that the following four conditions are satisfied:

(i) $p(x) = p(-x)$ for all x,

(ii) $p(x) \geq 0$ for all x,

(iii) $p(0) = 1$,

(iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

11. Let Q^+ denote the set of positive rational numbers. Show that there exists one and only one function $f\colon Q^+ \to Q^+$ satisfying the following conditions:

(i) If $0 < q < 1/2$ then $f(q) = 1 + f(q/(1-2q))$,

(ii) If $1 < q \leq 2$ then $f(q) = 1 + f(q-1)$,

(iii) $f(q) \cdot f(1/q) = 1$ for all $q \in Q^+$.

12. Let N denote the set of natural numbers. Let $\phi\colon N \to N$ be a **bijective** function and assume that there exists a finite limit

$$\lim_{n\to\infty} \frac{\phi(n)}{n} = L\,.$$

What are the possible values of L?

13. Prove that for any positive $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n}(x_i+y_i)^2}$$

holds.

14. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from another one. Prove that there is a town such that all remaining towns can be reached from it.

15. Noah has 8 species of animals to fit into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accomodation. Prove that there is a way to assign the animals to their cages so that each species shares with compatible species.

16. All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces?

17. Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that the diagonal AC is a diameter of the circle, while the other diagonal BD is as long as AB. The diagonals intesect at P. It is known that the length of PC is $2/5$. How long is the side CD?

18. Show that in a non-obtuse triangle the perimenter of the triangle is always greater than two times the diameter of the circumcircle.

19. Let C be a circle in plane. Let C_1 and C_2 be nonintersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2 touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

20. Let $a \leq b \leq c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that

$$p(p-c) = (p-a)(p-b) = S\,,$$

where S is the area of the triangle.