Baltic Way

15 November 2025, Riga, Latvia

Version: English

Time allowed: 4 hours and 30 minutes
Questions may be asked during the first 30 minutes.
Tools for writing and drawing are the only ones allowed.

Problem 1 Does there exist a sequence of integers $a_1, a_2, \ldots$ such that for each integer $d \neq 0$ there are exactly 2025 distinct pairs of indices (i, j) for which $a_i - a_j = d$?

Problem 2 Let $a_1, a_2, \ldots$ be a sequence of real numbers such that

$$\{a_n\}\lfloor a_{n+1}\rfloor = \lfloor a_n\rfloor\{a_{n+1}\}$$

for each positive integer n. Prove that there is a real number λ such that $\{a_m\}\lfloor a_m\rfloor = \lambda$ for infinitely many positive integers m.

Remark: For a real number x, $\lfloor x\rfloor$ denotes the greatest integer that does not exceed x, and $\{x\} = x - \lfloor x\rfloor$.

Problem 3 Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be a function that satisfies

$$f(x) + f(y) \geq f(x + y)$$

for all $x, y \in \mathbb{Q}$. Show that there exists an $\alpha \in \mathbb{R}$ such that $f(x) \geq \alpha x$ for all $x \in \mathbb{Q}$.

Problem 4 Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f(a-c) + f(b-d))\,(f(a) + f(b)) = f(ad - bc) + f\,(f(a) + f(b) - ac - bd)$$

for all real numbers a, b, c, d.

Problem 5 Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be positive real numbers satisfying

$$\frac{x_1^3 + x_2^3 + \cdots + x_n^3}{x_1 + x_2 + \cdots + x_n} = \frac{x_1^2 + x_n^2}{2}.$$

Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq n x_1 x_n.$$

Problem 6 Let $n \geq 3$ friends play dodgeball. Initially all players are free. When a free player A *knocks out* another free player B, A imprisons B and frees all players currently imprisoned by B. Two knock-outs cannot happen at the same time.

Find the least number of knock-outs after which it is possible that among every two players at least one has freed the other one.

Problem 7 Let n be an even positive integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. A set of n unit cubes that forms a $1 \times 1 \times n$ box is called a *needle*, which can have three orientations. Find the largest integer K, such that it is possible to select $3K$ needles, K in each orientation, such that no two of them share a unit cube.

Problem 8 A sequence of non-negative integers $a_1, \ldots, a_n$ is called *self-descriptive* if

$$a_k = |\{i \mid a_i \geq k\}|$$

for each $k \in \{1, \ldots, n\}$, i.e. a_k equals the number of elements of the sequence that are greater than or equal to k, for each $k \in \{1, \ldots, n\}$.

For each positive integer n, determine the number of self-descriptive sequences of length n.

Problem 9 A *Baltic polyiamond* is an n-gon with side lengths $n, n-1, \ldots, 2, 1$ in exactly this order, and all internal angles 120° or 240°. Prove that for every Baltic polyiamond, n is divisible by 6.

Problem 10 A positive integer with 9 digits is called *nice* if it contains each digit from 1 to 9. A number is called *wonderful* if it is nice and for each $k = 1, \ldots, 9$, it holds that the k-th digit in the number is equal to the position of k in the number. For example, 847296315 is wonderful:

- the first digit is 8, which is the position of 1 in the number,
- the second digit is 4, which is the position of 2 in the number,
- the third digit is 7, which is the position of 3 in the number, and so on.

Show that the number of wonderful numbers is even.

Problem 11 Let $A_1, \ldots, A_n$ be a sequence of points on the Euclidean plane. Some of the points may coincide, but the sequence contains at least two distinct points. A *Baltic Way* is a route defined by a permutation $B_1, \ldots, B_n$ of the points $A_1, \ldots, A_n$. The length of a Baltic Way is

$$|B_1B_2| + |B_2B_3| + \cdots + |B_{n-1}B_n| + |B_nB_1|,$$

where $|PQ|$ denotes the Euclidean distance between points P and Q.

Let K and L be the lengths of some two Baltic Ways on the same sequence of n points. For a given n, determine the maximum possible value of K/L over all possible sequences $A_1, \ldots, A_n$.

Problem 12 In an acute triangle ABC with $AB < AC$, the incentre is I and the circumcircle is ω. The line AI intersects the side BC at D. Let T be the point on ω such that $AT \parallel BC$. The line TI intersects ω again at P and the circumcircle of triangle APD again at Q. Prove that $AI = DQ$.

Problem 13 In a cyclic quadrilateral $ABCD$, the lines AB and CD intersect at E. A point P lies inside $ABCD$ and satisfies

$$\angle BAP = \angle PCB \quad \text{and} \quad \angle CBP = \angle PDC.$$

Prove that $PE \perp BC$.

Problem 14 In an acute triangle ABC with $AB < AC$, the circumcircle is ω and the orthocentre is H. The point P on major arc BC of ω satisfies $\angle PCB + \angle ACB = 90°$. The point R on segment AC satisfies $BR = CR$. Prove that $\angle HPR = \angle ACB$.

Problem 15 In an acute, scalene triangle ABC, the perpendicular bisector of BC intersects AC and AB in A_b and A_c, respectively. Define B_c, B_a, C_a and C_b similarly. Prove that the circumcentres of $ABC, A_bB_cC_a$, and $A_cB_aC_b$ are collinear.

Problem 16 Let a, b, c, n be positive integers satisfying $abc = 10^n - 1$. If $S(x)$ denotes the sum of digits of x in decimal notation, prove that

$$S(a) + S(b^2) + S(c^4) \geq \sqrt[3]{243n}.$$

Problem 17 A positive integer is called *brilliant* if its positive divisors can be partitioned into two sets with an equal number of elements and an equal sum of elements.

Find all possible values of the number of positive divisors of a brilliant number.

Problem 18 Find all functions $f\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$ such that $f(2) = 1$ and

$$\operatorname{lcm}(f(a+b), f(b)) \mid \operatorname{lcm}(a + f(b), b)$$

for all positive integers a, b.

Problem 19 There are 100 positive integers written on a blackboard. Anna plays a game. A move consists of choosing two integers a and b on the blackboard with the property that $a \mid b$, erasing them both and writing the integer b/a. She makes moves until there are no more valid moves, at which point the game ends.

Find the largest N such that for some initial state Anna can both:

- finish the game with only one number remaining, and
- finish the game with N numbers remaining.

Problem 20 Determine all positive integers n such that there exists a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$, where the numbers $a_1, 2a_2, \ldots, na_n$ are all pairwise distinct modulo n.