The Viking Battle - Part 1 2024

Problem 1 Let m and n be positive integers greater than 1. In each unit square of an $m \times n$ grid lies a coin with its tail-side up. A *move* consists of the following three steps:

1) select a 2×2 square S in the grid;
2) flip the coins in the top-left and bottom-right unit square of S;
3) flip the coin in either the top-right or bottom-left unit square of S.

Determine all pairs (m, n) for which it is possible that every coin shows head-side up after a finite number of moves.

Problem 2 Let ABC be a triangle with $AC > BC$. Let ω be the circumcircle of triangle ABC and let r be the radius of ω. The point P lies on the segment AC such that $BC = CP$ and the point S is the foot of the perpendicular from P to the line AB. Let the ray BP intersect ω again at D and let Q lie on the line SP such that $PQ = r$ and S, P and Q lie on the line in this order. Finally, let the line through A perpendicular to CQ intersect the line through B perpendicular to DQ at E.

Prove that E lies on ω.

Problem 3 Let $a_1 < a_2 < a_3 < \cdots$ be positive integers such that a_{k+1} divides $2(a_1 + a_2 + \cdots + a_k)$ for every $k \geq 1$. Suppose that for infinitely many primes p, there exists k such that p divides a_k.

Prove that for every positive integer n, there exists k such that n divides a_k.