The Viking Battle - Part 1 2024 - Solutions

Problem 1 Let m and n be positive integers greater than 1. In each unit square of an $m \times n$ grid lies a coin with its tail-side up. A *move* consists of the following three steps:

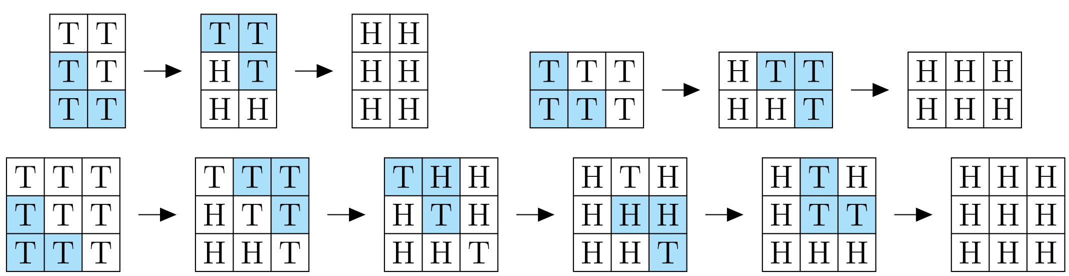
1) select a 2×2 square S in the grid;
2) flip the coins in the top-left and bottom-right unit square of S;
3) flip the coin in either the top-right or bottom-left unit square of S.

Determine all pairs (m, n) for which it is possible that every coin shows head-side up after a finite number of moves.

Solution to problem 1

Answer: All (m, n) where $3 \mid mn$.

First we show that if $3 \mid nm$, then it is possible. It is easy to see that any $m \times n$ grid where $3 \mid mn$ can be divided into 2×3, 3×2 and 3×3 rectangles. In a 2×3, a 3×2 and a 3×3 rectangle it is possible that every coin shows head-side up after a finite number of moves:



This shows the first part of the claim.

Now we prove that if $3 \nmid mn$, then it is not possible. Let (i, j) be the unit square in the ith row from the top, and the jth column from left. Label every (i, j) unit square the residue modulo 3 of $i + j - 2$:

0	1	2	0	$\cdots$
1	2	0	1	$\cdots$
2	0	1	2	$\cdots$
0	1	2	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Let $T(0)$, $T(1)$ and $T(2)$ be the number of 0, 1 and 2, respectively, in the grid with a coin that shows head-side up. At the beginning, we have $T(0) = T(1) = T(2) = 0$. Notice that in each move, each of $T(0)$, $T(1)$ and $T(2)$ is changed with ± 1, hence the parity of the pairwise differences of $T(0)$, $T(1)$ and $T(2)$ is invariant.

If all coins in the grid show head-side up, then

- if $m \equiv n \equiv 1 \pmod 3$, then $T(0) - 1 = T(1) = T(2) = \frac{mn-1}{3}$;
- if $nm \equiv 2 \pmod 3$, then $T(0) - 1 = T(1) - 1 = T(2) = \frac{mn-2}{3}$;
- if $m \equiv n \equiv 2 \pmod 3$, then $T(0) = T(1) - 1 = T(2) = \frac{mn-1}{3}$.

Hence in all scenarios where $3 \nmid mn$, the parity of at least one of the pairwise differences of $T(0)$, $T(1)$ and $T(2)$ would have to change in order to end with all coins head up, which is not possible.

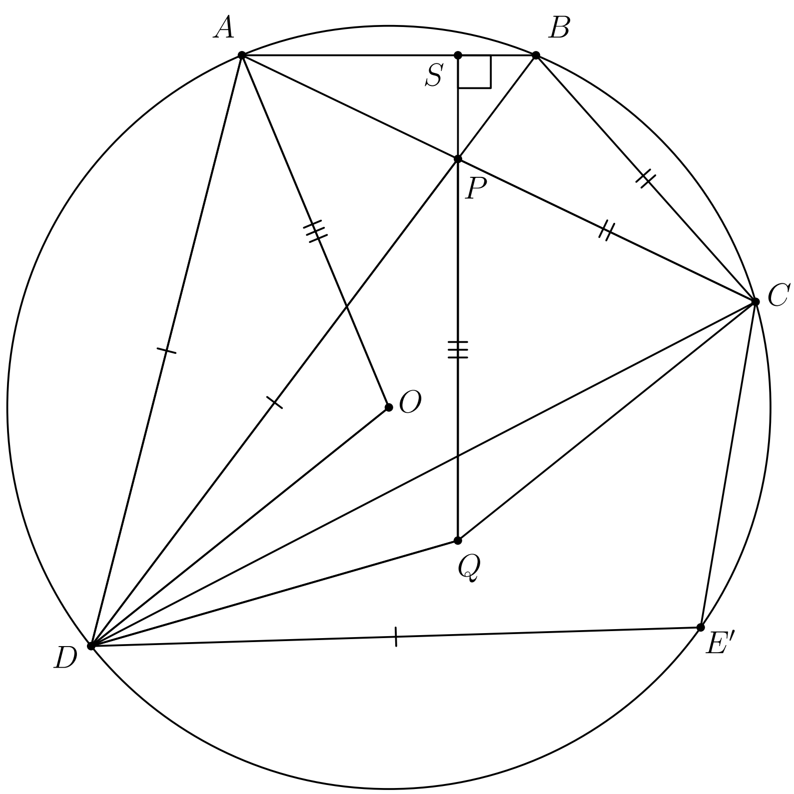
Problem 2 Let ABC be a triangle with $AC > BC$. Let ω be the circumcircle of triangle ABC and let r be the radius of ω. The point P lies on the segment AC such that $BC = CP$ and the point S is the foot of the perpendicular from P to the line AB. Let the ray BP intersect ω again at D and let Q lie on the line SP such that $PQ = r$ and S, P and Q lie on the line in this order. Finally, let the line through A perpendicular to CQ intersect the line through B perpendicular to DQ at E.

Prove that E lies on ω.

Solution to problem 2 First observe that since $CP = CB$ and $ABCD$ cyclic then

$$\angle DPA = \angle BPC = \angle CBP = \angle CBD = \angle CAD = \angle PAD,$$

and hence $DP = DA$. Thus there is a symmetry in the problem statement swapping $(A, D) \leftrightarrow (B, C)$.



Let O be the center of ω and let E' be the reflection of P in CD which by

$$\angle CE'D = \angle DPC = 180^\circ - \angle CPB = 180^\circ - \angle PBC = 180^\circ - \angle DBC$$

lies on ω. By construction we have $DE' = DP = DA$. We claim that $E = E'$: By the symmetry noted above, it suffices to prove that $BE' \perp DQ$ and then $AE' \perp CQ$

will follow by symmetry. We have $OA = PQ$, $AD = DP$ and

$$\angle DAO = 90^\circ - \frac{1}{2}\angle AOD = 90^\circ - \angle ABD = \angle BPS = \angle DPQ.$$

Hence $\triangle AOD \simeq \triangle PQD$. Thus

$$\begin{aligned}\angle QDB + \angle DBE' &= \angle ODA + \angle DAE' = \angle ODA + \angle AE'D \\ &= (90^\circ - \frac{1}{2}\angle AOD) + \angle AE'D = (90^\circ - \angle AE'D) + \angle AE'D = 90^\circ\end{aligned}$$

giving $BE' \perp DG$ as required.

Problem 3 Let $a_1 < a_2 < a_3 < \cdots$ be positive integers such that a_{k+1} divides $2(a_1 + a_2 + \cdots + a_k)$ for every $k \geq 1$. Suppose that for infinitely many primes p, there exists k such that p divides a_k.

Prove that for every positive integer n, there exists k such that n divides a_k.

Solution to problem 3

For every $k \geq 2$ define the quotient $b_k = 2(a_1 + a_2 + \cdots + a_{k-1})/a_k$. Since a_{k+1} divides $2(a_1 + a_2 + \cdots + a_k)$ for every $k \geq 1$, b_k is a positive integer.

First we prove that $b_{k+1} \leq b_k + 1$ for all $k \geq 2$. By subtracting $b_k a_k = 2(a_1 + a_2 + \cdots + a_{k-1})$ from $b_{k+1}a_{k+1} = 2(a_1 + a_2 + \cdots + a_k)$ we find that

$$b_{k+1}a_{k+1} = b_k a_k + 2a_k = (b_k + 2)a_k.$$

Since $a_{k+1} > a_k$, we get $b_{k+1} \leq b_k + 1$ for all $k \geq 2$.

Then we prove that the sequence (b_n) is unbounded. From $b_{k+1}a_{k+1} = (b_k + 2)a_k$, we know that $a_{k+1} \mid (b_k + 2)a_k$. Since this is true for all $k \geq 2$, recursively we get

$$a_{k+1} \mid a_k(b_k + 2) \mid a_{k-1}(b_{k-1} + 2)(b_k + 2) \mid \cdots \mid a_2 \prod_{i=2}^{k}(b_i + 2).$$

If the sequence (b_n) was bounded by B, then a_{k+1} would never be divisible by a prime p greater than $\max(a_2, B)$, which is not possible. Hence (b_n) is unbounded.

Since (b_n) is unbounded and $b_{k+1} \leq b_k + 1$ for all $k \geq 2$, then for all $n > b_2$ there exists a k such that $b_k = n - 1$ and $b_{k+1} = n$. Now

$$a_{k+1} = a_k \cdot \frac{b_k + 2}{b_{k+1}} = a_k \cdot \frac{n + 1}{n}.$$

Because n and $n + 1$ are coprime, this implies that a_k is divisible by n. If $n \leq b_2$, then chose an integer m such that $nm > b_2$. We know there exists a k such that nm divides a_k, and hence such that n divides a_k.