The Viking Battle - 2020

Problem 1 Let ABC be a triangle. The circle Γ passes through A, meets the segments AB and AC again at points D and E respectively, and intersects the segment BC at F and G, such that F lies between B and G. The tangent to the circle BDF at F and the tangent to the circle CEG at G meet at a point T. Suppose the points A and T are distinct. Prove that AT is parallel to BC.

Problem 2 We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \dots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Problem 3 On a flat plane in Camelot, King Arthur builds a labyrinth $\mathcal{L}$ consisting of n walls each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side meets a blue side, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners where two walls of different colours meet.

After Merlin paints the walls, Morgana then places som knights in the labyrinth. The knights can walk through doors, but cannot walk through walls. Let $k(\mathcal{L})$ be the largest number k such that no matter how Merlin paints the labyrinth $\mathcal{L}$, Morgana can always place k knights such that no two of them can ever meet.

For each n, what are all possible values for $k(\mathcal{L})$, where $\mathcal{L}$ is a labyrinth with n walls?