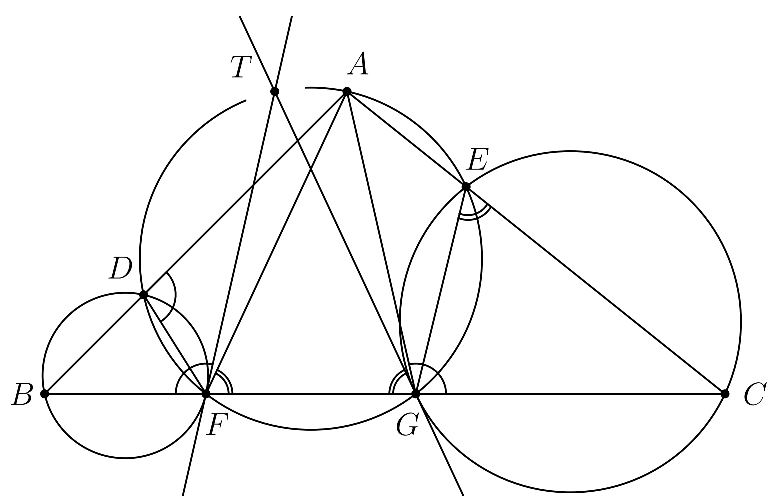
The Viking Battle - 2020
Solutions

Problem 1 Let ABC be a triangle. The circle Γ passes through A, meets the segments AB and AC again at points D and E respectively, and intersects the segment BC at F and G, such that F lies between B and G. The tangent to the circle BDF at F and the tangent to the circle CEG at G meet at a point T. Suppose the points A and T are distinct. Prove that AT is parallel to BC.

Solution Notice that $\angle TFB = \angle CGA$ because FT is tangent to circle BDF, and moreover $\angle FDA = \angle CGA$ because quadrilateral $ADFG$ is cyclic. Similarly, $\angle TGB = \angle GEC$ because GT is tangent to circle CEG, and $\angle GEC = \angle CFA$. Hence

$$\angle TFB = \angle CGA \quad \text{and} \quad \angle TGB = \angle CFA. \quad (*)$$



Triangles FGA and GFT have a common side FG, and by (*) their angles at F and T, respectively, are equal and hence AT is parallel to line $BFGC$.

Comment Alternatively, we can prove first that T lies on Γ.

Problem 2 We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \ldots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Solution The set $\mathbb{Z}$ is the only such rootiful set.

The set $\mathbb{Z}$ is clearly rootiful. We shall prove that any rootiful set S containing all the numbers $2^a - 2^b$ for positive integers a and b, must be all of $\mathbb{Z}$.

First, note that $0 = 2^1 - 2^1 \in S$ and $2 = 2^2 - 2^1 \in S$. Now, $-1 \in S$, since it is the root of $2x + 2$, and $1 \in S$ because it is a root of $2x^2 - x - 1$.Also if $n \in S$, then $-n \in S$ since $-n$ is the root of $x + n$. Hence it suffices to prove that all positive integers are in S.

First we prove that S contain a positive multiple of every positive integer n. Let $n = 2^\alpha \cdot u$ where α is a non-negative integer and u is odd. If $u = 1$ and $\alpha > 0$, then $n \mid 2^{\alpha+1} - 2^\alpha \in S$. If $u > 1$, then $u \mid 2^{\phi(u)} - 1 > 0$ and hence $2^\alpha \cdot u \mid 2^{\phi(u)+\alpha+1} - 2^{\alpha+1} \in S$, which proves the claim.

We will now prove by induction that all positive integers are in S. Suppose that $0, 1, \ldots, n-1$ are in S. Furthermore let $N \in S$ be a multiple of n. Consider the base-n expansion of N, $N = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$. Since $a_i \in \{0, 1, \ldots, n-1\}$ for $i = 0, 1, \ldots, k$, we have all $a_i \in S$. Furthermore $a_0 = 0$ since N is a multiple of n. Therefore

$$a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n - N = 0,$$

so n is the root of a polynomial with coefficients in S, and hence $n \in S$. This proves the induction step and completes the proof.

Problem 3 On a flat plane in Camelot, King Arthur builds a labyrinth $\mathcal{L}$ consisting of n walls each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side meets a blue side, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners where two walls of different colours meet.

After Merlin paints the walls, Morgana then places som knights in the labyrinth. The knights can walk through doors, but cannot walk through walls. Let $k(\mathcal{L})$ be the largest number k such that no matter how Merlin paints the labyrinth $\mathcal{L}$, Morgana can always place k knights such that no two of them can ever meet.

For each n, what are all possible values for $k(\mathcal{L})$, where $\mathcal{L}$ is a labyrinth with n walls?

Solution The only possible value of k is $k = n + 1$ no matter what shape the labyrinth is.

First we show by induction that the n walls divide the plane into $\binom{n+1}{2} + 1$ regions. The claim is true for $n = 0$ as when there are no walls the plane form a single region. When placing the n^{th} wall, it intersects each of the $n - 1$ other walls exactly once and hence splits each of n of the regions formed by those other walls into two regions. By the induction hypothesis this yields

$$\left(\binom{n}{2} + 1\right) + n = \binom{n+1}{2} + 1$$

regions, proving the claim.

Now let G be the graph with vertices given by the $\binom{n+1}{2} + 1$ regions, and with two regions connected by an edge if there is a door between them.

We now show that no matter how Merlin paints the n walls, Morgana can place at least $n + 1$ knights. No matter how the walls are painted, there are exactly $\binom{n}{2}$ intersection points, each of which correspond to a single edge in G. Consider adding the edges of G sequentially, and note that each edge reduces the number of connected components by at most one. Therefore the number of connected components of G is at least $\binom{n+1}{2} + 1 - \binom{n}{2} = n+1$. If Morgana places a knight in regions corresponding to different connected components of G, then no two knights can ever meet.

Now we give a construction showing that, no matter what shape the labyrinth is, Merlin can colour it such that there are exactly $n + 1$ connected components, allowing Morgana to place at most $n + 1$ knights.

First, we choose a coordinate system on the labyrinth so that none of the walls run due north-south, or due east-west. We then have Merlin paint the west face of each wall red, and the east face blue. We label the regions according to how many walls the region is on the east side of: the labels are integers between 0 and n.

We claim that, for each i, the regions labelled with i are connected by doors. First, we note that for each i with $0 \leq i \leq n$ there is a unique region labelled i which is unbounded to the north.

Now, consider a knight placed in some region with label i, and ask him to walk north (moving east or west by following the walls on the northern side of the region, as needed). This knight will never get stuck: Each region is convex, and so if it is bounded to the north, it has a single northernmost vertex with a door northwards to another region with label i.

Eventually it will reach a region which is unbounded to the north, which will be the unique such region with label i. Hence every region with label i are connected to each other.

As a result, there are exactly $n + 1$ connected components, and Morgana can place at most $n + 1$ knights.