The Viking Battle - Part 1 2019

Problem 1 Let $n \geq 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets such that one of these subsets has cardinality m and the sums of the elements in each subset are the same.

Problem 2 Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Problem 3 Given any set S of positive integers, show that at least one of the following two assertions holds

1) There exist distinct finite subsets F and G of S such that $\sum_{x \in F} \frac{1}{x} = \sum_{x \in G} \frac{1}{x}$.
2) There exists a positive rational number $r < 1$ such that $\sum_{x \in F} \frac{1}{x} \neq r$ for every finite subset F of S.

Problem 1 Let $n \geq 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets such that one of these subsets has cardinality m and the sums of the elements in each subset are the same.

Solution to problem 1 We prove that the following set fulfils the conditions:

$$S = \{3^k \mid k = 1, 2, \ldots, n-1\} \cup \{2 \cdot 3^k \mid k = 1, 2, \ldots, n-1\} \cup \{1, \frac{3^n + 9}{2} - 1\}.$$

It is easy to see that all these $2n$ numbers are different. The sum of the elements of S is

$$1 + \left(\frac{3^n + 9}{2} - 1\right) + \sum_{k=1}^{n-1}(3^k + 2 \cdot 3^k) = \frac{3^n + 9}{2} + \sum_{k=1}^{n-1} 3^{k+1} = \frac{3^n + 9}{2} + \frac{3^{n+1} - 9}{2} = 2 \cdot 3^n.$$

Hence we just have to show that for each $m = 2, 3, \ldots, n$ there exists an m-element subset A_m of S such that the sum of the elements of A_m is 3^n. Let

$$A_m = \{2 \cdot 3^k \mid k = n - (m-1), n - (m-2), \ldots, n-1\} \cup \{3^{n-m+1}\}.$$

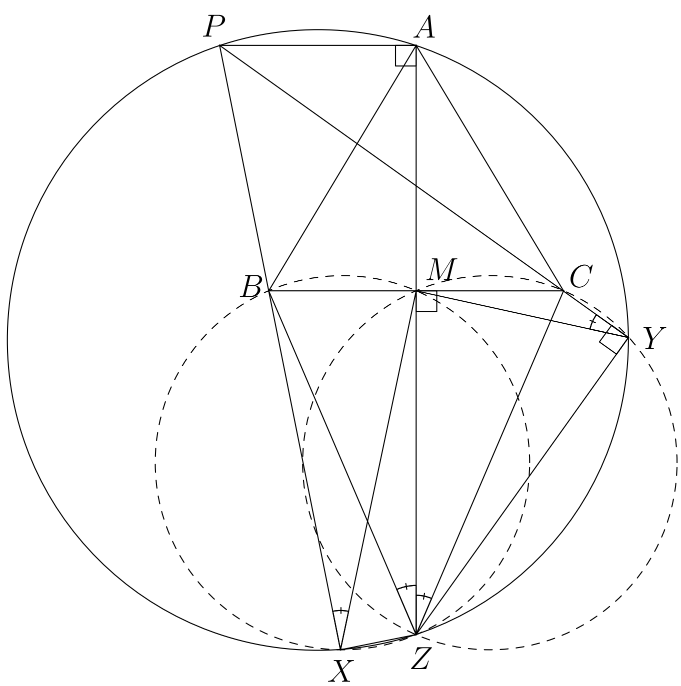
Clearly A_m has m elements. The sum of the elements of A_m is

$$3^{n-m+1} + \sum_{k=1}^{m-1} 2 \cdot 3^{n-k} = 3^{n-m+1} + 2 \cdot \frac{3^n - 3^{n-m+1}}{2} = 3^n$$

as required.

Problem 2 Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Solution to problem 2 Since $AB = AC$, we know that AM is perpendicular to BC, and hence PA is perpendicular to AM since it is parallel to BC. Let Z be the intersection of the line AM, and the line through Y perpendicular to PY. The circle with diameter PZ now goes through A and Y because $\angle PAZ = \angle PYZ = 90°$ by construction. To prove that $APXY$ is cyclic we just need to prove that X is on the circle with diameter PZ and hence that $\angle PXZ = 90°$.



By construction Z lies on the perpendicular bisector of BC. Hence the two circles with diametres BZ and CZ intersect at M. Since $\angle ZYC = 90°$, Y lies on the circle with diameter CZ. Hence

$$\angle BZM = \angle CZM = \angle CYM = \angle BXM$$

and thus X is on the circle BMZ. This proves that

$$\angle PXZ = \angle BXZ = 180° - \angle ZMB = 90°$$

as required.

Solution by inversion: Denoting by D and E the projections of M on the lines PB and PC, we have

$$\frac{DX}{DM} = \frac{EY}{EM}.$$

We invert in P and denote images by a prime. The previous relation then becomes

$$\frac{D'X'/(PD' \cdot PX')}{D'M'/(PD' \cdot PM')} = \frac{E'Y'/(PE' \cdot PY')}{E'M'/(PE' \cdot PM')},$$

or

$$\frac{D'X'}{PX' \cdot D'M'} = \frac{E'Y'}{PY' \cdot E'M'}.$$

Now notice that D, M, E, A and P are concyclic, so D', M', E' and A' are collinear. Further, because M and the point at infinity on the line BC divide segment BC harmonically, it follows by projection in P that M' and A' divide segment $D'E'$ harmonically. That is,

$$\frac{D'M'}{E'M'} = \frac{D'A'}{E'A'},$$

so the preceding equation becomes

$$\frac{D'X'}{PX' \cdot D'A'} = \frac{E'Y'}{PY' \cdot E'A'},$$

which, by the converse of Menelaos' theorem applied to $\triangle PD'E'$, implies (since X' and Y' are interior to the sides PD' and PE' while A' is exterior to the side $D'E'$) that X', Y' and A' are collinear. Then X, Y, A and P are concyclic.

Problem 3 Given any set S of positive integers, show that at least one of the following two assertions holds

1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F}\frac{1}{x}=\sum_{x\in G}\frac{1}{x}$.

2) There exists a positive rational number $r<1$ such that $\sum_{x\in F}\frac{1}{x}\neq r$ for every finite subset F of S.

Solution to problem 3 Assume by contradiction that neither of 1) and 2) holds. For every rational number $0\le r<1$ there exists a finite subset F_r of S such that $\sum_{x\in F_r}\frac{1}{x}=r$, and when $r=0$ let $F_0=\emptyset$.

We now prove that if $x\in S$ and q,r are two rational numbers $0<r<q<1$ such that $q-r=\frac{1}{x}$, then x is a member of F_q if and only if x is not a member of F_r. Assume that $x\in F_q$. Then

$$\sum_{y\in F_q\setminus\{x\}}\frac{1}{y}=q-\frac{1}{x}=r,$$

and hence $F_r=F_q\setminus\{x\}$. Conversely if x is not in F_r, then

$$\sum_{y\in F_r\cup\{x\}}\frac{1}{y}=r+\frac{1}{x}=q,$$

and hence $F_q=F_r\cup\{x\}$, and x is a member of F_q.

Consider now an element x of S and a positive rational number $r<1$. Let $n=\lfloor rx\rfloor$, and consider the sets $F_{r-\frac{n-k}{x}}$ for $k=0,\dots,n$. Notice first that x is not a member of $F_{r-\frac{n}{x}}$ since $r-\frac{n}{x}<\frac{1}{x}$. On the other hand $(r-\frac{n-1}{x})-(r-\frac{n}{x})=\frac{1}{x}$, so x must be a member of $F_{r-\frac{n-1}{x}}$. Since $(r-\frac{n-2}{x})-(r-\frac{n-1}{x})=\frac{1}{x}$ we deduce that x is not a member of $F_{r-\frac{n-2}{x}}$. By repeeding this argument we see that x is a member of $F_{r-\frac{n-k}{x}}$ when k is odd. Hence x is a memeber of F_r if n is odd.

Finally consider $F_{\frac{2}{3}}$. By the preceding $\lfloor\frac{2x}{3}\rfloor$ is odd for each x in $F_{\frac{2}{3}}$ so $\frac{2x}{3}$ is not an integer for any x in $F_{\frac{2}{3}}$. Since $F_{\frac{2}{3}}$ is finite, there exists a positive rational number ϵ such that $\lfloor\frac{2x-\epsilon}{3}\rfloor=\lfloor\frac{2x}{3}\rfloor$ for every $x\in F_{\frac{2}{3}}$, but this implies that $x\in F_{\frac{2}{3}-\epsilon}$ for every $x\in F_{\frac{2}{3}}$ and hence that $F_{\frac{2}{3}}\subseteq F_{\frac{2}{3}-\epsilon}$ which is imposible. Hence at least one of the two assertions holds.