The Viking Battle - Part 1 2018

Problem 1 Let q be a real number. A Viking has ten distinct real numbers written in his helmet, and he writes the following three lines of numbers in the sand.

- In the first line, the Viking writes down every number of the form $a - b$ where a and b are two (not necessarily distinct) numbers written in his helmet.
- In the second line, the Viking writes every number of the form qab where a and b are (not necessarily distinct) numbers written in the **first line**.
- In the third line, the Viking writes every number of the form $a^2 + b^2 - c^2 - d^2$ where a, b, c and d are (not necessarily distinct) numbers written in the **first line**.

Determine all values of q such that regardless of the numbers in the Viking's helmet, every number in the second line is also a number in the third line.

Problem 2 Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$ and $\angle CDE = \angle ABC$. Prove that the line from E perpendicular to BC, the line AC and the line BD are concurrent.

Problem 3 Let p be a prime number. Alice and Bob play the following game making moves alternately, and Alice has the first move. In each move, the player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then choose an element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The game ends after all the indices in the set $\{0, 1, 2, \ldots, p-1\}$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} a_{p-1}.$$

Alice wins if M is divisible by p, and Bob wins if it is not. Determine for each p which of the players has a winning strategy.