Solution to problem 1

Answer: $q = -2, 0, 2$.

First we show that the numbers $q = -2, 0, 2$ fulfil the conditions. The third line necessarily contains 0, so $q = 0$ works. For any two numbers a and b from the first line, write $a = x - y$ and $b = u - v$, where x, y, u and v are four (not necessarily distinct) numbers written in the Viking's helmet. Now

$$2ab = 2(x - y)(u - v) = (x - v)^2 + (y - u)^2 - (x - u)^2 - (y - v)^2,$$

and hence $q = 2$ works. By changing the sign of each side, we see that $q = -2$ also works.

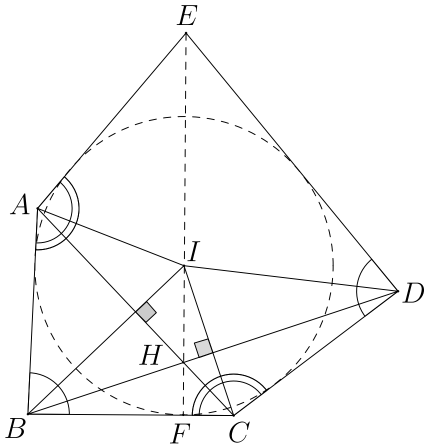
Now we prove that no other q works by looking at examples of ten numbers. Assume that $q \neq -2, 0, 2$. If the ten numbers in the helmet are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, then the numbers in the first line are $-9, -8, \ldots, 8, 9$. The second line then contains both q and $81q$, so the third line must contain both of them, and hence q is an integer since all numbers in the third line are integers. The largest number in the third line is $9^2 + 9^2 - 0^2 - 0^2 = 162$, and the smallest is $0^2 + 0^2 - 9^2 - 9^2 = -162$. Now $-162 \leq 81q \leq 162$, and hence $q = \pm 1$ since q is an integer and $q \neq -2, 0, 2$.

Now assume that $q = 1$ or $q = -1$. If the ten numbers in the helmet are $0, 1, 4, 8, 16, 20, 24, 28, 32$, then the first line contains the numbers ± 1 and ± 4, and so does the second line. If a is in the first line, then $a \not\equiv 2 \pmod 4$, and hence $a^2 \equiv 0$ or $a^2 \equiv 1 \pmod 8$. Consequently no number in the third line has residue 4 modulo 8, which is a contradiction since ± 4 are in line 2.

This shows that $q = -2, 0, 2$ are the only numbers that fulfil the conditions.

Solution to problem 2 The perpendicular bisector of AC passes trough B since triangle ABC is an isosceles triangle, and the perpendicular bisector of BD passes trough C since triangle BCD is an isosceles triangle. Let I be the intersection of these two perpendicular bisectors. Let H be the intersection of AC and BD. Now by construction H is the orthocenter of triangle BCI. It remains to prove that EI is perpendicular to BC. Lat F be the intersection of EI and BC and look at the two quadrilaterals $EABF$ and $EDCF$.

Since $\angle EAB = \angle FCD$ and $\angle ABF = \angle CDE$, if we prove that EI is the angle bisector of $\angle AED$, it will follow that $\angle BFE = \angle EFC$ and hence that EI i perpendicular to BC.



The lines BI and CI bisects $\angle ABC$ and $\angle BCD$, respectively. Since $IA = IC$, $IB = ID$ and $AB = BC = CD$, the triangles IAB, ICB and ICD are congruent. Hence $\angle IAB = \angle ICB = \frac{1}{2}\angle BCD = \frac{1}{2}\angle EAB$, so the line IA bisects $\angle EAB$. Similarly the line ID bisects $\angle CDE$. Finally IE bisects $\angle DEA$ since I lies on all the other four internal bisectors of the angles of the pentagon, and we are done.

Solution to problem 3

Answer: Alice has a winning strategy for every prime p.

If $p = 2$ or $p = 5$, then Alice chooses $a_0 = 0$ in the first move, and M becomes a multiple of 10 no matter what, and hence Alice wins.

Now assume that p is a prime and $p \neq 2, 5$. We say that a player makes the move (i, a_i) if she chooses the index i and the digit a_i. In the first move Alice makes the move $(p-1, 0)$, and then she pairs the indexes $0, 1, \ldots, p-2$ left after her first move in pairs $(k, k + \frac{p-1}{2})$, $k = 0, 1, \ldots, \frac{p-3}{2}$.

By Fermat's Little Theorem

$$(10^{\frac{p-1}{2}} - 1)(10^{\frac{p-1}{2}} + 1) = 10^{p-1} - 1 \equiv 1 - 1 = 0 \pmod{p},$$

and hence p divides either $10^{\frac{p-1}{2}} - 1$ or $10^{\frac{p-1}{2}} + 1$.

Case a: $10^{\frac{p-1}{2}} \equiv -1 \pmod{p}$. For each move (i, a_i) of Bob, Alice chooses (j, a_i) such that i and j are one of the pairs. Note that this is always possible. By following this strategy, in the end

$$\begin{aligned} M &= a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} a_{p-1} \\ &= \sum_{i=0}^{\frac{p-3}{2}} a_i (10^i + 10^{i+\frac{p-1}{2}}) = (1 + 10^{\frac{p-1}{2}}) \sum_{i=0}^{\frac{p-3}{2}} a_i \cdot 10^i \equiv 0 \pmod{p}, \end{aligned}$$

and hence Alice wins.

Case b: $10^{\frac{p-1}{2}} \equiv 1 \pmod{p}$. For each move (i, a_i) of Bob, Alice chooses $(j, 9 - a_i)$ such that i and j are one of the pairs. By following this strategy, in the end

$$\begin{aligned} M &= a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} a_{p-1} \\ &= \sum_{i=0}^{\frac{p-3}{2}} (a_i \cdot 10^i + (9 - a_i) 10^{i+\frac{p-1}{2}}) \equiv \sum_{i=0}^{\frac{p-3}{2}} (a_i \cdot 10^i + (9 - a_i) 10^i) \\ &= \sum_{i=0}^{\frac{p-3}{2}} 9 \cdot 10^i = 10^{\frac{p-1}{2}} - 1 \equiv 0 \pmod{p}, \end{aligned}$$

and hence Alice wins.