The Viking Battle - Part 1 2017
Version: English

Problem 1 Let n and k be positive integers where $k < n$. Peter and John play a game where they both know the rules. Peter chooses a secret n-digit binary string. Then he gives John a list of all n-digit binary strings that differ from his secret string in exactly k positions. (For example if $n = 3$, $k = 1$, and the secret string is 101, the list is 001, 111, 100). Now John has to guess the secret string.

What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Problem 2 For any positive integer k denote the sum of digits of k in its decimal representation by $S(k)$.

Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2017$ the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Problem 3 Let $B = (-1, 0)$ and $C(1, 0)$ be fixed points in the coordinate plane. A nonempty, bounded subset S of the plan is said to be a *viking* set if

(i) for any triangle $P_1P_2P_3$ there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which the triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar; and

(ii) there is a point T in S such that for every Q in S the segment TQ lies entirely in S.

Prove that there exist two distinct viking subsets S and S' of the set

$$\{(x, y) \mid x \geq 0, y \geq 0\}$$

such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (i) then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Solution to problem 1

Answer: If $k \neq \frac{n}{2}$, then the minimum number of guesses needed to guarantee the correct answer is 1, and if $k = \frac{n}{2}$, then the minimum number of guesses needed to guarantee the correct answer is 2.

If $k < \frac{n}{2}$, then instead we look at the string X' that is different from the secret string X at every position, and then the strings written to John differs from X' in exactly $n - k$ positions, and the task of guessing X is equivalent to the task of guessing X'. Hence wlog we can assume that $k \geq \frac{n}{2}$.

Let T be the set of all n-digit binary strings, and let S be the set of strings written on the list. Now let

$$Y \in T \backslash (S \cup X).$$

Then Y differs from X in exactly m positions, where m is a positive integer different from k. Assume wlog that Y differs from X at the first m positions, and look at $Z \in S$ that differs from X at the first k positions. Now Y and Z differ at $|k - m|$ positions. If $k \neq \frac{n}{2}$ or if $m \neq n$, then $|k - m| < k$ and hence John can rule out Y as the secret string. In the same way he can rule out every string in

$$T \backslash (S \cup X)$$

with one exception; he cannot rule out the string where $m = n$ in the case $k = \frac{n}{2}$. Thus if $k \neq \frac{n}{2}$ he can find X in one guess.

If $k = \frac{n}{2}$ and $m = n$ we have $|k - m| = k$. Hence Y is the string that differs from X at every position. This string is unique and it leads to the same list. Hence John is not able to deduce which of these strings is the secret one. In this case he can rule out all other strings but these two, and thus the minimum number of guesses needed to guarantee the correct answer is 2.

Solution to problem 2

Answer: $P(x) = c$, where $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, or $P(x) = x$.

Case 1. P is constant.

Let $P(x) = c$ where c is an integer. Then the condition becomes $S(c) = c$. This is true iff $c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Case 2. $\deg P = 1$.

We have the following observation. For any positive integers m and n, we have

$$S(m+n) \le S(m) + S(n), \quad \dagger$$

with equality if and only if there is no carry in the addition $m + n$.

Let $P(x) = ax + b$ for some integers a and b, $a \neq 0$. As $P(n)$ is positive for large n, we must have $a > 0$. The condition of the problem becomes $S(an+b) = aS(n)+b$ for all $n \ge 2017$. Setting $n = 2025$ and $n = 2020$ respectively, we get

$$S(2025a + b) - S(2020a + b) = (aS(2025) + b) - (aS(2020) + b) = 5a.$$

On the other hand, $\dagger$ implies

$$S(2025a + b) = S((2020a + b) + 5a)) \le S(2020a + b) + S(5a).$$

This gives $5a \le S(5a)$. As $a \ge 1$, this holds only when $a = 1$, since $S(n) < n$ for all $n > 9$. Hence $P(x) = x + b$ and $S(n + b) = S(n) + b$ for all $n \ge 2017$.

If $b > 0$, let k be a positive integer such that $10^k > b+2017$, and let $n = 10^k - b$. Notice that $n \ge 2017$. Now

$$S(n + b) = S(10^k) = 1 \quad \text{and} \quad S(n) + b \ge 1 + 1 = 2,$$

a contradiction.

If $b < 0$, let k be a positive integer such that $10^k > -b + 2017$, and let $n = 10^k$. Now

$$S(n + b) > 0 \quad \text{and} \quad S(n) + b = 1 + b \le 0,$$

a contradiction. Therefore, we conclude that $P(x) = x$, for which the condition is trivially satisfied.

Case 3. $\deg P > 1$.

Suppose $P(x) = a_d x^d + \cdots + a_1 x + a_0$, where $a_d \neq 0$ and $d > 1$. Since $P(x)$ is positive for large x, we know that $a_d > 0$. Consider $n = 10^k - 1$, $k \ge 4$. In this case the condition is

$$S(P(10^k - 1)) = P(9k)$$

for all $k \ge 4$. Now

$$P(9k) \ge \frac{1}{2} a_d (9k)^d$$

for sufficiently large k. We also know that $P(10^k - 1) \leq (10^k)^d \cdot 10^m = 10^{kd+m}$ for some konstant m and sufficiently large k, and hence

$$S(P(10^k - 1)) \leq 9(kd + m)$$

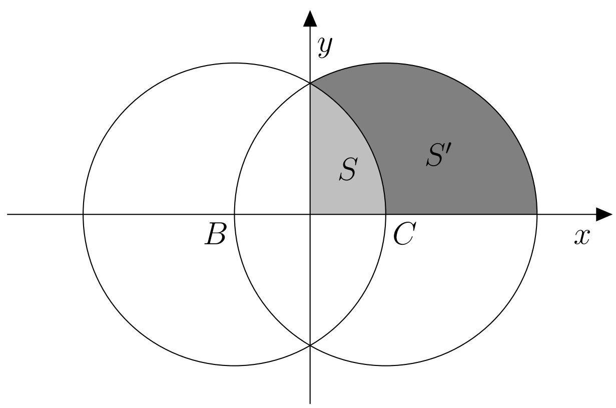
for sufficiently large k. This is a contradiction since $9(kd + m) < \frac{1}{2}a_d(9k)^d$ for sufficiently large k. Hence there are no solutions in this case.

Solution to problem 3

If in the similarity of $\triangle ABC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the longest side of $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, then we have $BC \geq AB \geq AC$. The condition $BC \geq AB$ is equivalent to $(x+1)^2 + y^2 \leq 4$ where $A = (x, y)$, while $AB \geq AC$ is trivially satisfied for any point in the first quadrant. Then we first define

$$S = \{(x, y) \mid (x+1)^2 + y^2 \leq 4, x \geq 0, y \geq 0\}.$$

Note that S is the intersection of a disk and the first quadrant, so it is bounded and convex, and we can choose any $T \in S$ to meet condition (ii). For any point A in S, the relation $BC \geq AB \geq AC$ always holds. Therefore, the point A in (i) is uniquely determined, while its existence is guaranteed by the above construction.



Next, if in the similarity of $\triangle A'BC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the second longest side of $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, then we have $A'B \geq BC \geq A'C$. The two inequalities are equivalent to $(x+1)^2 + y^2 \geq 4$ and $(x-1)^2 + y^2 \leq 4$ respectively, where $A' = (x, y)$. Then we define

$$S' = \{(x, y) \mid (x+1)^2 + y^2 \geq 4, (x-1)^2 + y^2 \leq 4, x \geq 0, y \geq 0\}.$$

The boundedness condition is satisfied while (i) can be argued as in the previous case. For (ii) note that S' contains points inside the disk $(x-1)^2 + y^2 \leq 4$ and outside the disk $(x+1)^2 + y^2 \leq 4$. This shows we can take $T' = (1, 2)$ in (ii), which is the topmost point of the circle $(x-1)^2 + y^2 = 4$.

It remains to check that the product $BA \cdot BA'$ is a constant. Suppose we are given a triangle $P_1P_2P_3$ with $P_1P_2 \geq P_2P_3 \geq P_3P_1$. By the similarities, we have

$$BA = BC \cdot \frac{P_2P_3}{P_1P_2} \quad \text{and} \quad BA' = BC \cdot \frac{P_1P_2}{P_2P_3}.$$

Thus $BA \cdot BA' = BC^2 = 4$, which is certainly independent of the triangle $P_1P_2P_3$.