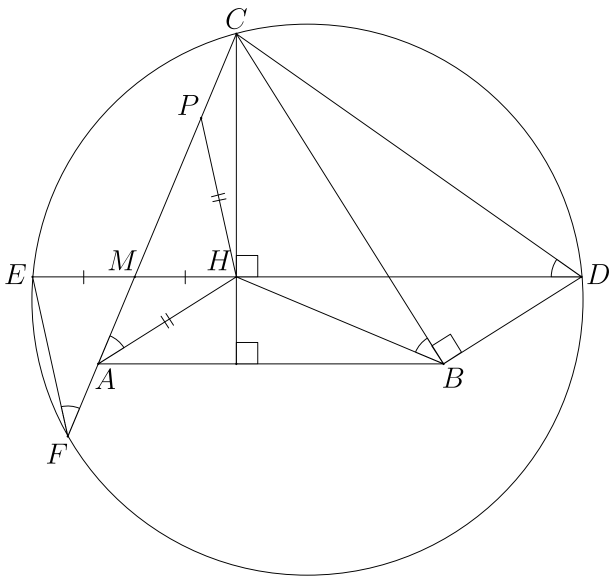
Solution to problem 1 Since $HD \parallel AB$ and $BD \parallel AH$, we have $BD \perp BC$ and $CH \perp DH$. Hence the quadrilateral $BDCH$ is cyclic. Since H is the orthocenter of the triangle ABC, wa have $\angle HAC = 90^\circ - \angle ACB = \angle CBH$. Using that $BDCH$ and $CDFE$ are cyclic quadrilaterals we get

$$\angle CFE = \angle CDH = \angle CBH = \angle HAC.$$

Let M be the intersection of AC and DH and hence the midpoint of EH by construction. Let $P \neq A$ be the point on the line AC such that $AH = HP$. Then $\angle MFE = \angle HAC = \angle MPH$. Since $MFE = \angle MPH$, $\angle FME = \angle HMP$, and $EM = MH$, the triangles EMF and HMP are congruent, and thus $EF = HP = AH$.



Comment Instead of introducing the point P, one can complete the solution by using the law of sines in the triangles EFM and AMH, yielding:

$$\frac{EF}{EM} = \frac{\sin \angle EMF}{\sin \angle MFE} = \frac{\sin \angle AMH}{\sin \angle HAM} = \frac{AH}{MH} = \frac{AH}{EM}.$$

Solution to problem 2 From the constraint of the problem we see that

$$\frac{k}{a_{k+1}} \le \frac{a_k^2 + k - 1}{a_k} = a_k + \frac{k-1}{a_k},$$

and so

$$a_k \ge \frac{k}{a_{k+1}} - \frac{k-1}{a_k}.$$

Summing up the above inequality for $k = 1, 2, \ldots, m$, we obtain

$$a_1 + a_2 + \cdots + a_m \ge \left(\frac{1}{a_2} - \frac{0}{a_1}\right) + \left(\frac{2}{a_3} - \frac{1}{a_2}\right) + \cdots + \left(\frac{m}{a_{m+1}} - \frac{m-1}{a_m}\right) = \frac{m}{a_{m+1}}.$$

Now we prove the problem statement by induction on n. The case $n = 2$ can be done applying the constraint for $k = 1$:

$$a_1 + a_2 \ge a_1 + \frac{1}{a_1} \ge 2.$$

For the induction step, assume that the statement is true for some $n \ge 2$. If $a_{n+1} \ge 1$, then the induction hypothesis yields

$$(a_1 + \cdots + a_n) + a_{n+1} \ge n + 1.$$

Otherwise, if $a_{n+1} < 1$ then

$$(a_1 + \cdots + a_n) + a_{n+1} \ge \frac{n}{a_{n+1}} + a_{n+1} = \frac{n-1}{a_{n+1}} + \left(\frac{1}{a_{n+1}} + a_{n+1}\right) > (n-1) + 2 = n + 1.$$

That completes the solution.

Solution to problem 3

Answer: The game ends in a draw when $n = 1, 2, 4, 6$, otherwise B wins.

Firstly, we show that b wins whenever $n \neq 1, 2, 4, 6$. For this purpose, we provide a strategy which guarantees that B can always make a move after A's move, and also guarantees that the game does not end in a draw.

We begin by proving that B can always move: By symmetry we can assume that A starts by choosing a number not exceeding $\frac{n+1}{2}$. Then B chooses n. After A has made the k^{th} move where $k \geq 2$, we now prove that B can also make a move. Let S be the set of all the k numbers chosen by A so far. Then the set $\{1, 2, \ldots, n\} \backslash S$ consists of k or $k + 1$ "contiguous components". Since B has only chosen $k - 1$ numbers, there is at least one component of $\{1, 2, \ldots, n\} \backslash S$ consisting of numbers not yet picked by B. Hence B can choose a number from this component.

Case 1: Assume that $n \geq 3$ is odd. The only way the game can end in a draw, is if A picks all the odd numbers. But this situation cannot happen since B picks n as the first number.

Case 2: Assume that $n \geq 8$ is even: Since B picks n, the only way the game can end in a draw is if A picks all the odd numbers. After the second move of A, there is at least one odd number less than $n - 1$ left, and hence B can prevent a draw by picking an odd number in the second move.

Case $n = 1, 2, 4$. If $n = 1, 2$ the game obviously ends in a draw. If $n = 4$, the only way A can prevent loosing, is by picking 1 or 4. Assume wlog that A picks 1. Then B has to pick 4 not to loose, and hence A can pick 3 and B is left with 2.

Case $n = 6$. The above shows that B gets at least a draw. On the other hand, A may also get at least a draw in the following way: First A picks 1. After that B picks a number b. If $b = 6$, then A reserves 5 for the last move, and picks 3. If $b = 4, 5$, then A reserves $b+1$ for the last move, and picks 3. If $b = 2, 3$, then A reserves $b+1$ for the last move, and picks 6.