The Viking Battle - Part 1 2015
Version: Icelandic

Dæmi 1 Látum $n \geq 2$ vera heiltölu og látum A_n vera mengið

$$A_n = \{2^n - 2^k | k \in \mathbb{Z}, 0 \leq k < n\}.$$

Ákvarðið stærstu heiltölu M_n sem ekki er hægt að rita sem summu einnar eða fleiri ekki nauðsynlega ólíkra staka úr A_n.

Dæmi 2 Skilgreinum fallið $f : (0,1) \to (0,1)$ með

$$f(x) = \begin{cases} x + \frac{1}{2} & , \ x < \frac{1}{2} \\ x^2 & , \ x \geq \frac{1}{2} \end{cases}$$

Látum a_0 og b_0 vera tvær rauntölur þannig að $0 < a_0 < b_0 < 1$. Við skilgreinum runurnar a_n og b_n með $a_n = f(a_{n-1})$ og $b_n = f(b_{n-1})$ fyrir öll $n = 1, 2, 3, \ldots$.

Sýnið að til er jákvæð heiltal n þannig að

$$(a_n - a_{n-1}) \cdot (b_n - b_{n-1}) < 0.$$

Dæmi 3 Látum Ω og O vera umhring og ummiðju hvasshyrnds þríhyrnings ABC með $AB > BC$. Helmingalína hornins $\angle ABC$ sker Ω í $M \neq B$. Látum Γ vera hring með þvermál BM. Helmingalínur hornanna $\angle AOB$ og $\angle BOC$ skera Γ í punktunum P og Q í þeirri röð. Punkturinn R er valinn á línunni PQ þannig að $BR = MR$. Sannið að $BR \parallel AC$. (Hér gerum við alltaf ráð fyrir að helmingalínur horna séu hálflínur.)

Solution to problem 1
Answer: $M_n = (n-2)2^n + 1$.

Part 1. First we prove that every integer greater than $(n-2)2^n + 1$ can be represented as such a sum. This is achieved by induction on n.

For $n = 2$, the set $A_n = \{2, 3\}$. Every positive integer m except 1 can be represented as a sum of elements of A_n: as $m = 2 + 2 + \cdots + 2$ if m is even, and as $m = 3 + 2 + 2 + \cdots + 2$ if m is odd.

Now consider some $n > 2$ and assume the induction hypothesis holds for $n - 1$. Take an integer $m > (n-2)2^n + 1$. If m is even, then

$$\frac{m}{2} > (n-2)2^{n-1} > ((n-1)-2)2^{n-1} + 1.$$

Hence by the induction hypothesis

$$\frac{m}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i, with $0 \le k_i < n - 1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}),$$

giving us the desired representation as a sum of elements of A_n. If m is odd, we consider

$$\frac{m - (2^n - 1)}{2} > \frac{(n-2)2^n + 1 - (2^n - 1)}{2} = (n-3)2^{n-1} + 1.$$

By the induction hypothesis there is a representation of the form

$$\frac{m - (2^n - 1)}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i, with $0 \le k_i < n - 1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}) + (2^n - 1),$$

giving us the desired representation of m once again.

Part 2. It remains to prove that there is no representation of $M_n = (n-2)2^n + 1$. Let N be the smallest positive integer that satisfies $N \equiv 1 \pmod{2^n}$, and which can be represented as a sum of elements of A_n. Consider the representation of N, i.e.

$$N = (2^n - 2^{k_1}) + (2^n - 2^{k_2}) + \cdots + (2^n - 2^{k_r}),$$

where $0 \le k_1, k_2, \ldots, k_r < n$. If $k_i = k_j = n-1$, then we can simply remove these two terms from the sum to get a representation for $N - 2(2^n - 2^{n-1}) = N - 2^n$ as a sum of elements of A_n, which contradicts our choice of N. If $k_i = k_j = k < n-1$, replace the two terms by $2^n - 2^{k+1}$, which is also an element of A_n, to get a representation for $N - 2(2^n - 2^k) + 2^n - 2^{k+1} = N - 2^n$. This is a contradiction once again. Therefor, all k_i have to be distinct, which means that

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \leq 2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1.$$

On the other hand

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \equiv -\Big((2^n - 2^{k_1}) + (2^n - 2^{k_2}) + \cdots + (2^n - 2^{k_r})\Big) = -N \equiv -1 \pmod{2^n}$$

Thus we must have $2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} = 2^n - 1$, which is only possible if each element of $\{0, 1, 2, \ldots, n-1\}$ occurs as one of the k_i. This gives us

$$N = n2^n - (2^0 + 2^1 + \cdots + 2^{n-1}) = (n-1)2^n + 1.$$

In particular this means that $(n-2)2^n + 1$ cannot be represented as a sum of elements of A_n.

Solution to problem 2
Note that

$$f(x) - x = \frac{1}{2} > 0 \text{ if } x < \frac{1}{2}$$
$$f(x) - x = x^2 - x < 0 \text{ if } x \geq \frac{1}{2}.$$

We consider the interval $(0, 1)$ divided into the two subintervals $I_1 = (0, \frac{1}{2})$ and $I_2 = [\frac{1}{2}, 1)$. The inequality

$$0 > (a_n - a_{n-1}) \cdot (b_n - b_{n-1}) = (f(a_{n-1}) - a_{n-1})(f(b_{n-1} - b_{n-1})$$

holds if and only if a_{n-1} and b_{n-1} lie in distinct subintervals.

Let us now assume, to the contrary, that a_k and b_k always lie in the same subinterval. Consider the distance $d_k = |a_k - b_k|$. If both a_k and b_k lie in I_1, then

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left| a_k + \frac{1}{2} - \left(b_k + \frac{1}{2} \right) \right| = d_k.$$

If, on the other hand, a_k and b_k both lie in I_2, then $a_k + b_k \geq \frac{1}{2} + \frac{1}{2} + d_k = 1 + d_k$, which implies

$$d_{k+1} = |a_{k+1} - b_{k+1}| = |a_k^2 - b_k^2| = |(a_k - b_k)(a_k + b_k)| \geq d_k(1 + d_k).$$

This means that the difference d_k is non-decreasing, and particular $d_k \geq d_0 > 0$ for all k.

If a_k and b_k lie in I_2, then

$$d_{k+2} \geq d_{k+1} \geq d_k(1 + d_k) \geq d_k(1 + d_0).$$

If a_k and b_k lie in I_1, then a_{k+1} and b_{k+1} both lie in I_2, and so we have

$$d_{k+2} \geq d_{k+1}(1 + d_{k+1}) \geq d_{k+1}(1 + d_0) \geq d_k(1 + d_0).$$

In either case, $d_{k+2} \geq d_k(1 + d_0)$, and inductively we get

$$d_{2m} \geq d_0(1 + d_0)^m.$$

For sufficiently large m, the right-hand side is greater than 1, a contradiction. Thus there must be a positive integer n such that a_{n-1} and b_{n-1} do not lie in the same subinterval, which proves the desired statement.

Solution to problem 3

Let K be the midpoint of BC, i.e. the centre of Γ. Notice that $AB \neq BC$ implies that $K \neq O$. Clearly the lines OM and OK are perpendicular bisectors of AC and BM, respectively. Therefore, R is the intersection point of PQ and OK.

Let N be the second point of intersection of Γ with the line OM. Hence $BN \parallel AC$, and it suffices to prove that BN passes trough R. Our plan for doing this is to interpret the lines BN, OK and PQ as the radical axes of three appropriate circles.

Let ω be the circle with diameter BO. Since $\angle BNO = \angle BKO = 90°$, the points N and K lie on ω.

Next we show that the points O, K, P, and Q are concyclic. To this end, let D and E be the midpoints of BC and AB, respectively. By our assumption of triangle ABC, the points B, E, O, K, and D lie on ω is this order. It follows that $\angle EOR = \angle EBK = \angle KBD = \angle KOD$, so the line KO externally bisects the angle POQ. Since the point K is the centre of Γ, it also lies on the perpendicular bisector of PQ. So K coincides with the midpoint of the arc POQ of the circumcircle γ of triangle POQ.

Thus the lines OK, BN, and PQ are pairwise radical axis of the circles ω, γ and Γ. Hence they are concurrent at R, as required.