

The Viking Battle - Part 1 2015

Version: Finnish

Tehtävä 1 Olkoon ≥ 2 kokonaisluku, ja olkoon A_n joukko

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z}, 0 \leq k < n\}.$$

Selvitä suurin kokonaisluku M_n , jota ei voi kirjoittaa yhden tai useamman joukon A_n alkion summana. (Alkioiden ei välttämättä tarvitse olla erisuuria.)

Tehtävä 2 Määritellään funktio $f : (0, 1) \rightarrow (0, 1)$ asettamalla

$$f(x) = \begin{cases} x + \frac{1}{2} & , x < \frac{1}{2} \\ x^2 & , x \geq \frac{1}{2} \end{cases}$$

Olkoot a_0 ja b_0 kaksi reaalilukua, joille $0 < a_0 < b_0 < 1$. Määritellään jonot a_n ja b_n asettamalla $a_n = f(a_{n-1})$ ja $b_n = f(b_{n-1})$ kaikille $n = 1, 2, 3, \dots$

Osoita, että on olemassa positiivinen kokonaisluku n , jolle

$$(a_n - a_{n-1}) \cdot (b_n - b_{n-1}) < 0.$$

Tehtävä 3 Oletetaan, että teräväkärkisessä kolmiossa ABC pätee $AB > BC$. Olkoon Ω kolmion ABC ympäripiirretty ympyrä, ja olkoon O ympyrän Ω keskipiste. Kulman $\angle ABC$ kulmanpuolittaja leikkaa ympyrän Ω pisteessä $M \neq B$. Olkoon Γ se ympyrä, jonka eräs halkaisija on BM . Kulman $\angle AOB$ puolittaja leikkaa ympyrää Γ pisteessä P , ja kulman $\angle BOC$ puolittaja leikkaa ympyrää Γ pisteessä Q . Suoralta PQ on valittu piste R siten, että $BR = MR$. Osoita, että $BR \parallel AC$. (Tässä oletamme aina, että kulmanpuolittaja on puolisuora.)

Solution to problem 1

Answer: $M_n = (n - 2)2^n + 1$.

Part 1. First we prove that every integer greater than $(n - 2)2^n + 1$ can be represented as such a sum. This is achieved by induction on n .

For $n = 2$, the set $A_n = \{2, 3\}$. Every positive integer m except 1 can be represented as a sum of elements of A_n : as $m = 2 + 2 + \dots + 2$ if m is even, and as $m = 3 + 2 + 2 + \dots + 2$ if m is odd.

Now consider some $n > 2$ and assume the induction hypothesis holds for $n - 1$. Take an integer $m > (n - 2)2^n + 1$. If m is even, then

$$\frac{m}{2} > (n - 2)2^{n-1} > ((n - 1) - 2)2^{n-1} + 1.$$

Hence by the induction hypothesis

$$\frac{m}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \dots + (2^{n-1} - 2^{k_r})$$

for some k_i , with $0 \leq k_i < n - 1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \dots + (2^n - 2^{k_r+1}),$$

giving us the desired representation as a sum of elements of A_n . If m is odd, we consider

$$\frac{m - (2^n - 1)}{2} > \frac{(n - 2)2^n + 1 - (2^n - 1)}{2} = (n - 3)2^{n-1} + 1.$$

By the induction hypothesis there is a representation of the form

$$\frac{m - (2^n - 1)}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \dots + (2^{n-1} - 2^{k_r})$$

for some k_i , with $0 \leq k_i < n - 1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \dots + (2^n - 2^{k_r+1}) + (2^n - 1),$$

giving us the desired representation of m once again.

Part 2. It remains to prove that there is no representation of $M_n = (n - 2)2^n + 1$. Let N be the smallest positive integer that satisfies $N \equiv 1 \pmod{2^n}$, and which can be represented as a sum of elements of A_n . Consider the representation of N , i.e.

$$N = (2^n - 2^{k_1}) + (2^n - 2^{k_2}) + \dots + (2^n - 2^{k_r}),$$

where $0 \leq k_1, k_2, \dots, k_r < n$. If $k_i = k_j = n - 1$, then we can simply remove these two terms from the sum to get a representation for $N - 2(2^n - 2^{n-1}) = N - 2^n$ as a sum

of elements of A_n , which contradicts our choice of N . If $k_i = k_j = k < n - 1$, replace the two terms by $2^n - 2^{k+1}$, which is also an element of A_n , to get a representation for $N - 2(2^n - 2^k) + 2^n - 2^{k+1} = N - 2^n$. This is a contradiction once again. Therefore, all k_i have to be distinct, which means that

$$2^{k_1} + 2^{k_2} + \dots + 2^{k_r} \leq 2^0 + 2^1 + \dots + 2^{n-1} = 2^n - 1.$$

On the other hand

$$2^{k_1} + 2^{k_2} + \dots + 2^{k_r} \equiv -\left((2^n - 2^{k_1}) + (2^n - 2^{k_2}) + \dots + (2^n - 2^{k_r})\right) = -N \equiv -1 \pmod{2^n}$$

Thus we must have $2^{k_1} + 2^{k_2} + \dots + 2^{k_r} = 2^n - 1$, which is only possible if each element of $\{0, 1, 2, \dots, n - 1\}$ occurs as one of the k_i . This gives us

$$N = n2^n - (2^0 + 2^1 + \dots + 2^{n-1}) = (n - 1)2^n + 1.$$

In particular this means that $(n - 2)2^n + 1$ cannot be represented as a sum of elements of A_n .

Solution to problem 2

Note that

$$\begin{aligned} f(x) - x &= \frac{1}{2} > 0 \text{ if } x < \frac{1}{2} \\ f(x) - x &= x^2 - x < 0 \text{ if } x \geq \frac{1}{2}. \end{aligned}$$

We consider the interval $(0, 1)$ divided into the two subintervals $I_1 = (0, \frac{1}{2})$ and $I_2 = [\frac{1}{2}, 1)$. The inequality

$$0 > (a_n - a_{n-1}) \cdot (b_n - b_{n-1}) = (f(a_{n-1}) - a_{n-1})(f(b_{n-1}) - b_{n-1})$$

holds if and only if a_{n-1} and b_{n-1} lie in distinct subintervals.

Let us now assume, to the contrary, that a_k and b_k always lie in the same subinterval. Consider the distance $d_k = |a_k - b_k|$. If both a_k and b_k lie in I_1 , then

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left| a_k + \frac{1}{2} - \left(b_k + \frac{1}{2} \right) \right| = d_k.$$

If, on the other hand, a_k and b_k both lie in I_2 , then $a_k + b_k \geq \frac{1}{2} + \frac{1}{2} + d_k = 1 + d_k$, which implies

$$d_{k+1} = |a_{k+1} - b_{k+1}| = |a_k^2 - b_k^2| = |(a_k - b_k)(a_k + b_k)| \geq d_k(1 + d_k).$$

This means that the difference d_k is non-decreasing, and particular $d_k \geq d_0 > 0$ for all k .

If a_k and b_k lie in I_2 , then

$$d_{k+2} \geq d_{k+1} \geq d_k(1 + d_k) \geq d_k(1 + d_0).$$

If a_k and b_k lie in I_1 , then a_{k+1} and b_{k+1} both lie in I_2 , and so we have

$$d_{k+2} \geq d_{k+1}(1 + d_{k+1}) \geq d_{k+1}(1 + d_0) \geq d_k(1 + d_0).$$

In either case, $d_{k+2} \geq d_k(1 + d_0)$, and inductively we get

$$d_{2m} \geq d_0(1 + d_0)^m.$$

For sufficiently large m , the right-hand side is greater than 1, a contradiction. Thus there must be a positive integer n such that a_{n-1} and b_{n-1} do not lie in the same subinterval, which proves the desired statement.

Solution to problem 3

Let K be the midpoint of BC , i.e. the centre of Γ . Notice that $AB \neq BC$ implies that $K \neq O$. Clearly the lines OM and OK are perpendicular bisectors of AC and BM , respectively. Therefore, R is the intersection point of PQ and OK .

Let N be the second point of intersection of Γ with the line OM . Hence $BN \parallel AC$, and it suffices to prove that BN passes through R . Our plan for doing this is to interpret the lines BN , OK and PQ as the radical axes of three appropriate circles.

Let ω be the circle with diameter BO . Since $\angle BNO = \angle BKO = 90^\circ$, the points N and K lie on ω .

Next we show that the points O , K , P , and Q are concyclic. To this end, let D and E be the midpoints of BC and AB , respectively. By our assumption of triangle ABC , the points B , E , O , K , and D lie on ω in this order. It follows that $\angle EOR = \angle EBK = \angle KBD = \angle KOD$, so the line KO externally bisects the angle POQ . Since the point K is the centre of Γ , it also lies on the perpendicular bisector of PQ . So K coincides with the midpoint of the arc POQ of the circumcircle γ of triangle POQ .

Thus the lines OK , BN , and PQ are pairwise radical axes of the circles ω , γ and Γ . Hence they are concurrent at R , as required.

