The Viking Battle - Part 1 2014
Problems and solutions

Problem 1 Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Solution 1 Setting $(m, n) = (f(1), 1)$ gives

$$f(1)^2 + f(1) \mid f(1)f(f(1)) + 1,$$

and hence $f(1) \mid 1$. Thus $f(1) = 1$.

When $(m, n) = (m, 1)$ and $(m, n) = (1, m)$ we get

$$m^2 + 1 \mid mf(m) + 1 \quad \text{and} \quad 1 + f(n) \mid 1 + n$$

for all positive integers n and m. This proves that $f(m) \geq m$ and $f(n) \leq n$. Hence the only possible function is $f(n) \equiv n$.

It is easy to see that the function $f(n) \equiv n$ is a solution.

Solution 2 Setting $(m, n) = (2, 2)$ gives

$$4 + f(2) \mid 2f(2) + 2.$$

Since $2f(2) + 2 < 2(4 + f(2))$, we must have $2f(2) + 2 = 4 + f(2)$, so $f(2) = 2$. Now when $m = 2$ we have $4 + f(n) \mid 4 + n$ which implies that $f(n) \leq n$ for all n.

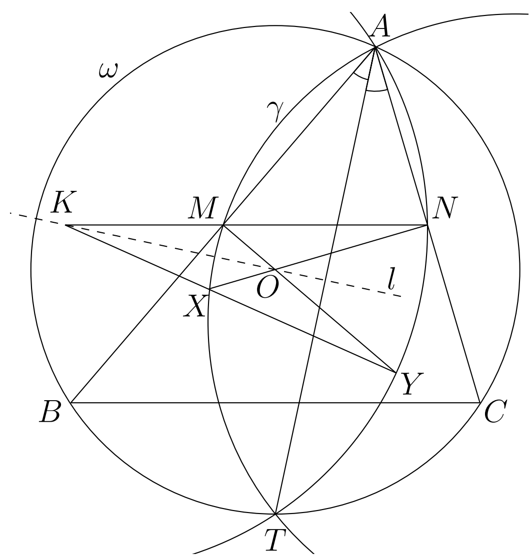
Further $n = m$ tells that

$$n^2 + f(n) \mid nf(n) + n$$

and hence $n^2 + f(n) \leq nf(n) + n$ which we rewrites as $f(n) \geq \frac{n^2-n}{n-1} = n$ for all $n > 1$. Hence $f(n) = n$ for all $n > 1$, and it is easy to see that $f(1) = 1$ as well.

Problem 2 Let ω be the circumcircle of triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively. Assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

Solution Let O be the center of ω, thus O is the intersection of MY and NX. Let l be the perpendicular bisector of AT and notice that l passes through O. Denote by r the reflection about l. We want to prove that $r(M) = X$ and $r(N) = Y$ since this proves that the intersection point K of MN and XY is on l and hence $KA = KT$.



Since AT is the angle bisector of $\angle BAC$, the line $r(AB)$ is parallel to AC. Since $OM \perp AB$ and $ON \perp AC$, this means that the line $r(OM)$ is parallel to the line ON and passes through O, so $r(OM) = ON$. Finally the circumcircle γ of the triangle AMT is symmetric about l, so $r(\gamma) = \gamma$. Thus the point M maps to the common point of ON and the arc AMT of γ, that is $r(M) = X$. Similarly $r(N) = X$, and then we are done.

Problem 3 A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons are *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in his lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

Solution Let us consider a graph with the imons as vertices, and two imons being connected by an edge if and only if they are entangled. A *proper colouring* of a graph G is a colouring of its vertices in several colours so that every two connected vertices have different colours. Let $c(G)$ be the minimal number of colours in a proper colouring of G. We want to prove that if a graph G with $c(G) = n$, $n > 1$, then one may perform a sequence of operations on G resulting in a graph G' with $c(G') < n$. By applying this several times, we get a graph that has a proper colouring of one colour, and hence a graph with no edges which was to be proved.

Now assume that G is a graph with $c(G) = n$, $n > 1$. Let us repeatedly apply operation (i) to any vertex with odd degree as long as it is possible. This results in a graph G_1 with $c(G_1) \leq n$. If $c(G_1) < n$ we are done. If not we colour the vertices of G_1 in n colours $1, 2, \ldots, n$ such that it is a proper colouring. We then apply operation (ii) to this graph and get a new graph G_2. We colour the vertex I' in colour $k+1 \pmod{n}$ where k is the colour of the vertex I. Then two connected original vertices still have different colours, and so do their two connected copies. Since $n > 1$ the vertices I and I' have different colours as well. Thus $c(G_2) = n$. Now all the vertices in G_2 have an odd degree. If we look at all the vertices of colour n, no two of these are connected. Hence by applying (i) several times we can delete all these vertices and get a new graph G' with $c(G') \leq n - 1 < n$. This ends the proof.