The Viking Battle - Part 1 2013
Problems and solutions

Problem 1 Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y+1, x)$ or $(x-1, x)$. Prove that she can perform only finitely many such iterations.

Solution We solve the more general problem where x and y are not necessarily adjacent in the row and the operation consists in replacing (x, y) with (z, x), where z is any number in the interval $y \leq z \leq x$. Since for $x > y$ we have $y \leq y+1, x-1 \leq x$, the given problem is a special case of this one.

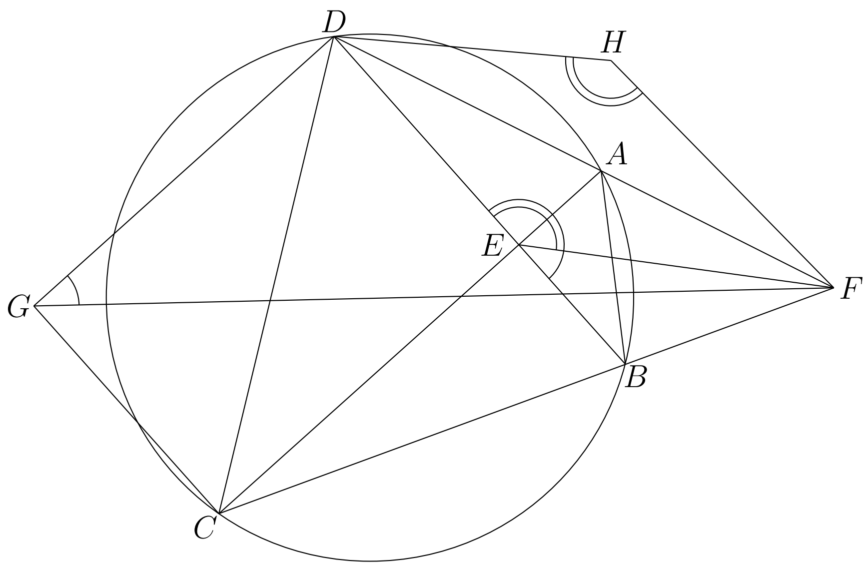
First note that the allowed operation does not change the maximum M of the sequence, and consider the sum

$$S = a_1 + 2a_2 + \cdots + na_n,$$

where a_i is the ith number in the row. In each step, the rightmost number increases by $x - y$ and the leftmost one decreases by at most this difference. Since the rightmost number has the highest weight in the sum S, this sum therefore increases. Since S cannot exceed $(1 + 2 + \cdots + n)M$, the process then stops after a finite number of operations.

Problem 2 Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, and G are concyclic.

Solution Since $\angle FAB = \angle FCD$, a transformation composed of a homothety with centre F and the reflection in the bisector of $\angle AFB$ maps segment AB to segment CD. Since $\triangle ABE \sim \triangle DCE \sim \triangle CDG$, this transformation maps E to G and we have $\angle FGD = \angle FEB = 180° - \angle FED = 180° - \angle FHD$. Hence the assertion follows.



Problem 3 Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Solution First note that x divides $2012 \cdot 2 = 2^3 \cdot 503$. If 503 divides x then the right-hand side has to be divisible by 503^3 and hence 503^2 divides $xyz + 2$. This is impossible since 503 divides x. Now x divides 2^3, and $x = 2^m$, $m \in \{0, 1, 2, 3\}$. If $m \geq 2$, then 2^6 divides the left-hand side but not the right-hand side, hence $x = 1$ or $x = 2$. This reduces the equation to

$$y^3 + z^3 = 2012(yz + 2) \text{ if } x = 1,$$
$$y^3 + z^3 = 503(yz + 1) \text{ if } x = 2.$$

In both cases 503 divides $y^3 + z^3$, and hence $y^3 \equiv (-z)^3 \pmod{503}$. If 503 does not divide z and y, this leads to $(y(-z)^{-1})^3 \equiv 1 \pmod{503}$. By Fermat's little theorem $(y(-z)^{-1})^{502} \equiv 1 \pmod{503}$ and hence $y(-z)^{-1} \equiv (y(-z)^{-1})^{\gcd(3,502)} \equiv 1 \pmod{503}$, thus $y \equiv -z \pmod{503}$. If 503 divides z it also divides y, and hence in both cases $y + z$ is divisible by 503.

Let $y + z = 503k$, $k \geq 1$. In view of $y^3 + z^3 = (y + z)((z - y)^2 + yz)$ the equation is reduced to

$$k(z-y)^2 + (k-4)yz = 8 \text{ if } x = 1,$$
$$k(z-y)^2 + (k-1)yz = 1 \text{ if } x = 2.$$

If $x = 1$ we have $(k-4)yz \leq 8$, which implies $k \leq 4$ since $z \geq \frac{503}{2}$. If we look at the original equation, it is clear that in this case $y^3 + z^3$ is even, and hence that $k \cdot 503$ is even too, meaning that k is even. Thus $k = 2$ or $k = 4$. Clearly the reduced equation has no solutions for $k = 4$. If $k = 2$ then $(z-y)^2 - yz = 4$ and hence $2^2 \cdot 503^2 - 4 = (z+y)^2 - 4 = 5yz$. However $2^2 \cdot 503^2 - 4$ is not divisible by 5. Therefore there are no solutions in the case $x = 1$.

If $x = 2$ then $0 \leq (k-1)yz \leq 1$, and since $z \geq \frac{503}{2}$ we have $k = 1$. Now $z - y = 1$ and $z + y = 503$. This leads to $y = 251$ and $z = 252$. It is easy to see that $(2, 251, 252)$ is a solution.

In summary the triple $(2, 251, 252)$ is the only solution.