

Den 30:e Nordiska matematiktävlingen

Tisdag, 5 april 2016

Skrivtid: 4 timmar. Varje problem är värt 7 poäng. Enda tillåtna hjälpmedel är skriv- och ritdon.

Problem 1

Bestäm alla följder av icke-negativa heltal a_1, \dots, a_{2016} , alla mindre än eller lika med 2016, som uppfyller $i + j \mid ia_i + ja_j$ för alla $i, j \in \{1, 2, \dots, 2016\}$.

Problem 2

Låt $ABCD$ vara en inskriven fyrhörning sådan att $AB = AD$ och $AB + BC = CD$. Bestäm $\angle CDA$.

Problem 3

Bestäm alla $a \in \mathbb{R}$, för vilka det finns en funktion $f : \mathbb{R} \rightarrow \mathbb{R}$, sådan att

- (i) $f(f(x)) = f(x) + x$, för alla $x \in \mathbb{R}$,
- (ii) $f(f(x) - x) = f(x) + ax$, för alla $x \in \mathbb{R}$.

Problem 4

Kung George har bestämt sig för att koppla samman de 1680 öarna i riket med broar. Olyckligtvis kommer rebellrörelsen att spränga två broar efter att alla broar blivit byggda, men det kan inte vara två broar från samma ö.

Vilket är det minsta antalet broar kungen måste bygga för att vara säker på att det fortfarande går att färdas över broar mellan vilka två av de 1680 öarna som helst efter att rebellrörelsen sprängt två broar?

The 30th Nordic Mathematical Contest

Tuesday, April 5, 2016

Solutions

Problem 1

Determine all sequences of non-negative integers a_1, \dots, a_{2016} all less than or equal to 2016 satisfying $i + j \mid ia_i + ja_j$ for all $i, j \in \{1, 2, \dots, 2016\}$.

Solution Answer: All constant sequences of non-negative integers.

The condition rewrites to $i + j \mid i(a_i - a_j)$. Since $2k - 1$ and k are coprime, we see that $2k - 1 \mid a_k - a_{k-1}$. Thus if $2k - 1 > 2016$, then $a_k = a_{k-1}$ since a_k and a_{k-1} are non-negative and at most 2016. All together $a_{1009} = a_{1010} = \dots = a_{2016}$.

If $i < 1009$ we know that i is coprime to one of the number $2016, 2015, \dots, 2017 - i$ say j . Then $i + j \mid a_i - a_j$ and since $i + j > 2016$ we conclude as before that $a_i = a_j = a_{2016}$. So any such sequence is constant.

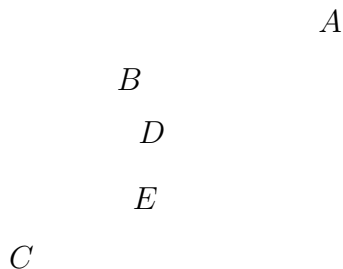
Problem 2

Let $ABCD$ be a cyclic quadrilateral satisfying $AB = AD$ and $AB + BC = CD$.

Determine $\angle CDA$.

Solution 2 Answer: $\angle CDA = 60^\circ$.

Choose the point E on the segment CD such that $DE = AD$. Then $CE = CD - AD = CD - AB = BC$, and hence the triangle CEB is isosceles.



Since $AB = AD$ then $\angle BCA = \angle ACD$. This shows that CA is the bisector of $\angle BCD = \angle BCE$. In an isosceles triangle, the bisector of the apex angle is also the perpendicular bisector of the base. Hence A is on the perpendicular bisector of BE , and $AE = AB = AD = DE$. This shows that triangle AED is equilateral, and thus $\angle CDA = 60^\circ$.

Problem 3

Find all $a \in \mathbb{R}$ for which there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$, such that

(i) $f(f(x)) = f(x) + x$, for all $x \in \mathbb{R}$,

(ii) $f(f(x) - x) = f(x) + ax$, for all $x \in \mathbb{R}$.

Solution 3 Answer: $a = \frac{1 \pm \sqrt{5}}{2}$.

From (i) we get $f(f(f(x)) - f(x)) = f(x)$. On the other hand (ii) gives

$$f(f(f(x)) - f(x)) = f(f(x)) + af(x).$$

Thus we have $(1 - a)f(x) = f(f(x))$. Now it follows by (i) that $(1 - a)f(x) = f(x) + x$, and hence $f(x) = -\frac{1}{a}x$, since $a = 0$ obviously does not give a solution.

We now need to check whether (i) and (ii) hold for this function for some values of a and all real x . We have

$$f(f(x)) = -\frac{1}{a}f(x) = \frac{1}{a^2}x, \text{ and } f(x) + x = -\frac{1}{a}x + x = \frac{a-1}{a}x.$$

Thus (i) will hold for all real x iff $\frac{1}{a^2} = \frac{a-1}{a}$, i.e. iff $a = \frac{1 \pm \sqrt{5}}{2}$. For these values of a we have

$$f(f(x) - x) = -\frac{1}{a}(f(x) - x) = -\frac{1}{a}\left(-\frac{1}{a}x - x\right) = \left(\frac{1}{a^2} + \frac{1}{a}\right)x = \frac{a+1}{a^2}x = x,$$

and

$$f(x) + ax = -\frac{1}{a}x + ax = \frac{a^2 - 1}{a}x = x,$$

so that for these two values of a both (i) and (ii) hold for all real x . Thus the values of a such that there exists a function f with the desired properties are $a = \frac{1 \pm \sqrt{5}}{2}$.

Problem 4

King George has decided to connect the 1680 islands in his kingdom by bridges. Unfortunately the rebel movement will destroy two bridges after all the bridges have been built, but not two bridges from the same island.

What is the minimal number of bridges the King has to build in order to make sure that it is still possible to travel by bridges between any two of the 1680 islands after the rebel movement has destroyed two bridges?

Solution 4 Answer: 2016

An island cannot be connected with just one bridge, since this bridge could be destroyed. Two islands, each with only two bridges, cannot be connected by a bridge since they would be isolated if the rebel movement destroys the two bridges from these islands not connecting the two.

Let x be the number of islands with exactly two bridges. Then $1680 - x$ islands have at least three bridges each. Let B be the number of bridges. Now $B \geq 2x$ and $2B \geq 2x + 3(1680 - x) = 5040 - x$. Hence $2B \geq \max(4x, 5040 - x) \geq 4 \cdot \frac{5040}{5} = 4032$, and thus $B \geq 2016$.

It is possible to construct an example with exactly 2016 bridges: Take 672 of the islands and number them $0, 1, 2, \dots, 671$. Connect island number i with the islands numbered $i - 1$, $i + 1$ and $i + 336$ (modulo 672). This gives 1008 bridges. We now have a circular path of 672 bridges: $0 - 1 - 2 - \dots - 671 - 0$. If one of these 672 bridges are destroyed, the 672 islands are still connected. If two of these bridges are destroyed, the path is broken into two parts. Let i be an island on the shortest path (if they have the same length, just pick a random one). Then island $i + 336$ (modulo 672) must be on the other part of the path, and the bridge connecting these two islands will connect the two paths. Hence no matter which two bridges the rebel movement destroys, it is possible to travel between any of the 672 islands.

Now for every of the 1008 bridges above, replace it with two bridges with a new island between the two. This increases the number of bridges to 2016 and the number of islands to $672 + 1008 = 1680$ completing the construction. Since the rebel movement does not destroy two bridges from the same island, the same argument as above shows that with this construction it is possible to travel between any of the 1680 islands after the destruction of the two bridges.