The 29th Nordic Mathematical Contest

Tuesday, March 24, 2015

Problem 1.

Let ABC be a triangle and Γ the circle with diameter AB. The bisectors of $\angle BAC$ and $\angle ABC$ intersect Γ (also) at D and E, respectively. The incircle of ABC meets BC and AC at F and G, respectively. Prove that D, E, F and G are collinear.

Problem 2.

Find the primes p, q, r, given that one of the numbers pqr and $p+q+r$ is 101 times the other.

Problem 3.

Let $n > 1$ and $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with n real roots (counted with multiplicity). Let the polynomial q be defined by

$$q(x) = \prod_{j=1}^{2015} p(x+j).$$

We know that $p(2015) = 2015$. Prove that q has at least 1970 different roots $r_1, \ldots, r_{1970}$ such that $|r_j| < 2015$ for all $j = 1, \ldots, 1970$.

Problem 4.

An encyclopedia consists of 2000 numbered volumes. The volumes are stacked in order with number 1 on top and 2000 in the bottom. One may perform two operations with the stack:

(i) For n even, one may take the top n volumes and put them in the bottom of the stack without changing the order.

(ii) For n odd, one may take the top n volumes, turn the order around and put them on top of the stack again.

How many different permutations of the volumes can be obtained by using these two operations repeatedly?

Time allowed: 4 hours.

Each problem is worth 7 points.

Only writing and drawing tools are allowed.