NMC 2013: Marking scheme

Problem 1

1. rewriting in general $\lfloor n + \sqrt{n} + \frac{1}{2} \rfloor$ without the use of the Floor function for $(k + \frac{1}{2})^2 \leq n < (k+1)^2$ and $k^2 \leq n < (k + \frac{1}{2})^2$, only involving k and the difference between n and k^2 or n and $(k+1)^2$, but not n: 1 point
2. same as above, but with the bounds $k^2 + k + 1 \leq n < (k+1)^2$ and $k^2 \leq n \leq k^2 + k$: 2 points
3. conjecturing that $\lfloor n + \sqrt{n} + \frac{1}{2} \rfloor$ is never a perfect square for $n \geq 1$, and that therefore there is exactly one square in the sequence: 1 point
4. conjecturing the correct values $a_{2m} = 1 + m^2$ and $a_{2m+1} = 1 + m(m+1)$ and concluding that there is exactly one square in the sequence: 3 points

Item 3 is additive with each of 1 and 2, but not 4.

Problem 2

1. for conjecturing the correct lower bound of $n - 2$: 0 point
2. for conjecturing the correct lower bound of $n-2$ and constructing a correct (realisable) "final standings table" for all n (i.e. number of W/D/L and scores, but not the results of individual matches): 1 point
3. for conjecturing the correct lower bound of $n - 2$ and giving appropriate full tournament result examples for $n = 4, 5$: 1 point
4. for showing the correct lower bound of $n - 2$: 2 points

The items are additive, a full solution is worth 5 points

Problem 3
"continued fraction" approach:

1. relating the binary form of the index k to a modified finite continued fraction form of q_k: 2 points
2. proving that this modified finite continued fraction form is unique or that it yields all rational numbers: +1 point

inductive proofs:

1. showing that all integers can be found exactly once: 1 point
2. showing that all inverses of integers can be found exactly once: 1 point
3. showing all rationals appear at least once/at most once: 2/2 points

Items 1, 2 and 3 are additive with an upper bound of 3 points total for an incomplete solution.
An essentially complete solution with minor gaps is worth 4 points.

Problem 4

1. proving one direction only, with the argumentation inappropriate for the other direction: 2 points
2. proving and claiming one direction only, with the argumentation appropriate for the other direction: 3 points
3. claiming a complete solution but equivalence not being stated consequently: 4 points
4. full solution 5 points

   to all of the above

5. failing to explicitly indicate where non-degeneracy comes into play: -1 point