Solutions

Problem 1

When $a_0, a_1, \ldots, a_{1000}$ denote digits, can the sum of the 1001-digit numbers $a_0 a_1 \ldots a_{1000}$ and $a_{1000} a_{999} \ldots a_0$ have odd digits only?

Solution. The answer is no.

The following diagram illustrates the calculation of the sum digit by digit.

	a_0	a_1	$\ldots$	a_i	$\ldots$	a_{500}	$\ldots$	a_{1000-i}	$\ldots$	a_{998}	a_{999}	a_{1000}
	a_{1000}	a_{999}	$\ldots$	a_{1000-i}	$\ldots$	a_{500}	$\ldots$	a_i	$\ldots$	a_2	a_1	a_0
s_{1001}	s_{1000}	s_{999}	$\ldots$	s_{1000-i}	$\ldots$	s_{500}	$\ldots$	s_i	$\ldots$	s_2	s_1	s_0

Thus s_i are the digits of the sum. The digit s_{1001} may be absent. We call column i the column in the diagram with the digit s_i.

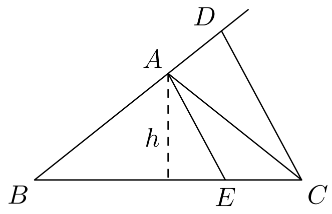
Assume that s_i is odd for $i = 0, 1, \ldots, 1000$. By induction on i we prove that $a_{2i} + a_{1000-2i}$ is odd for $i = 0, 1, \ldots, 250$. This implies that $a_{2\cdot 250} + a_{1000-2\cdot 250} = 2a_{500}$ is odd, which is a contradiction. Here is the proof: Since s_0 is odd, $a_0 + a_{1000}$ is odd, so the statement is true for $i = 0$. Assume that $a_{2i} + a_{1000-2i}$ is odd for some $i = 0, 1, \ldots, 249$. Since $s_{1000-2i}$ is odd and $a_{2i} + a_{1000-2i}$ is odd, there is no carry in column $1000 - 2i$, so $a_{2i+1} + a_{1000-(2i+1)} \leq 9$. But then because s_{2i+1} is odd and $a_{2i+1} + a_{1000-(2i+1)} \leq 9$, there is no carry in column $2i + 2$. Hence $a_{2i+2} + a_{1000-(2i+2)}$ is odd because s_{2i+2} is odd. This completes the induction step.

Problem 2

In a triangle ABC assume $AB = AC$, and let D and E be points on the extension of segment BA beyond A and on the segment BC, respectively, such that the lines CD and AE are parallel. Prove $CD \geq \dfrac{4h}{BC} CE$, where h is the height from A in triangle ABC. When does equality hold?

Solution. From $AE \parallel CD$ we get $CD/AE = BC/BE$, whence

$$CD = \frac{AE \cdot BC}{BE \cdot CE} CE\,. \qquad (1)$$



Since h is the shortest distance of A from the line BC, we have $AE \geq h$ with equality when E is the foot of the height, which, due to $AB = AC$, is the midpoint of BC. Futhermore $BE \cdot CE \leq ((BE + CE)/2)^2 = (BC/2)^2$ with equality when $BE = CE$, which is equivalent to E being once more the midpoint of BC. Combining these results we get from (1) the requested inequality with equality when E is the midpoint of BC.

Problem 3

Find all functions f such that

$$f(f(x)+y)=f(x^2-y)+4yf(x)$$

for all real numbers x and y.

Solution. Substituting $y=(x^2-f(x))/2$ yields $2(x^2-f(x))f(x)=0$, so for each x either $f(x)=0$ or $f(x)=x^2$. If $f(x)=0$ one gets $f(y)=f(x^2-y)$. In particular $f(y)=f(-y)$. If $f(y)\neq 0$, the last but one equation can be written $y^2=(x^2-y)^2$. Now, due to $f(y)=f(-y)$, if $f(y)\neq 0$ also $f(-y)\neq 0$, so we have $y^2=(x^2+y)^2$ as well. Adding these equations gives $2x^4=0$, or $x=0$. In conclusion, if $f(y)\neq 0$ for some y then $f(x)=0$ implies $x=0$. Thus either $f(x)=0$ for all x or $f(x)=x^2$ for all x. It is easily verified that both these functions satisfy the given equation.

Problem 4

Show that for any integer $n\geq 2$ the sum of the fractions $\dfrac{1}{ab}$, where a and b are relatively prime positive integers such that $a<b\leq n$ and $a+b>n$, equals $\dfrac{1}{2}$. (Integers a and b are called *relatively prime* if the greatest common divisor of a and b is 1.)

Solution. We prove this by induction. First observe that the statement holds for $n=2$ because $a=1$ and $b=2$ are the only numbers which satify the conditions. Next we show that increasing n by 1 does not change the sum, so it remains equal to $1/2$. To that end it suffices to show that the sum of the terms removed from the sum equals the sum of the new terms added. All the terms in the sum for $n-1$ remain in the sum for n except the fractions $1/ab$ with a and b relatively prime, $0<a<b\leq n$ and $a+b=n$. On the other hand the new fractions added to the sum for n have the form $1/an$ with $0<a<n$. So it suffices to show

$$\sum_{\substack{0<a<n/2\\ \gcd(a,n-a)=1}}\frac{1}{a(n-a)}=\sum_{\substack{0<a<n\\ \gcd(a,n)=1}}\frac{1}{an}\,.$$

Note that the terms in the sum on the right hand side can be grouped into pairs

$$\frac{1}{an}+\frac{1}{(n-a)n}=\frac{(n-a)+a}{a(n-a)n}=\frac{1}{a(n-a)}$$

because $\gcd(a,n)=\gcd(n-a,n)$, so either both these terms are in the sum or none of them is. No term is left out because if n is even and greater than 2 then $\gcd(n/2,n)=n/2>1$. So the right hand side is given by

$$\sum_{\substack{0<a<n\\ \gcd(a,n)=1}}\frac{1}{an}=\sum_{\substack{0<a<n/2\\ \gcd(a,n)=1}}\frac{1}{a(n-a)}=\sum_{\substack{0<a<n/2\\ \gcd(a,n-a)=1}}\frac{1}{a(n-a)}\,,$$

which is what we had to prove.