The 23rd Nordic Mathematical Contest

Thursday April 2, 2009

English version with solutions

Time allowed is 4 hours. Each problem is worth 5 points. The only permitted aids are writing and drawing tools.

Problem 1

A point P is chosen in an arbitrary triangle. Three lines are drawn through P which are parallel to the sides of the triangle. The lines divide the triangle into three smaller triangles and three parallelograms. Let f be the ratio between the total area of the three smaller triangles and the area of the given triangle. Show that $f \geq \frac{1}{3}$ and determine those points P for which $f = \frac{1}{3}$.

Problem 2

On a faded piece of paper it is possible, with some effort, to discern the following:

$$(x^2 + x + a)(x^{15} - \ldots) = x^{17} + x^{13} + x^5 - 90x^4 + x - 90.$$

Some parts have got lost, partly the constant term of the first factor of the left side, partly the main part of the other factor. It would be possible to restore the polynomial forming the other factor, but we restrict ourselves to asking the question: What is the value of the constant term a? We assume that all polynomials in the statement above have only integer coefficients.

Problem 3

The integers 1, 2, 3, 4 and 5 are written on a blackboard. It is allowed to wipe out two integers a and b and replace them with $a + b$ and ab. Is it possible, by repeating this procedure, to reach a situation where three of the five integers on the blackboard are 2009?

Problem 4

There are 32 competitors in a tournament. No two of them are equal in playing strength, and in a one against one match the better one always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.

Solution 1

Because the sides are parallel, all the three smaller triangles are similar to the given triangle. Let h be the length of a height in the given triangle and g be the length of the belonging base. The corresponding numbers in the three smaller triangles are h_i and g_i for $i = 1, 2, 3$. Because of the similarity there exist a number

$$k = \frac{h}{g} = \frac{h_1}{g_1} = \frac{h_2}{g_2} = \frac{h_3}{g_3}.$$
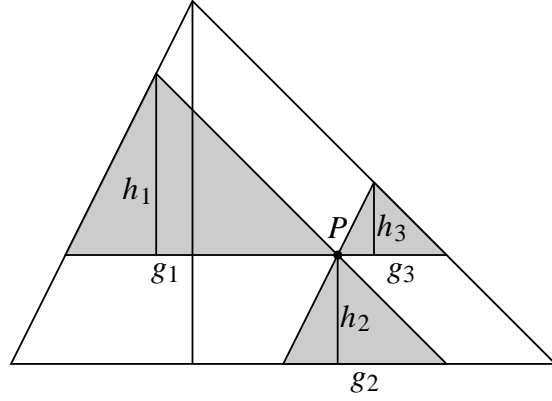
Also it is easily seen that $g = g_1 + g_2 + g_3$ and $h = h_1 + h_2 + h_3$. We have

$$f \geq \frac{1}{3} \quad \Leftrightarrow \quad \frac{1}{2} hg \leq 3 \cdot \frac{1}{2}(h_1 g_1 + h_2 g_2 + h_3 g_3)$$
$$\Leftrightarrow \quad (h_1 + h_2 + h_3)(g_1 + g_2 + g_3) \leq 3(h_1 g_1 + h_2 g_2 + h_3 g_3)$$

The last inequality is Chebyshev's inequality (because $h_i \leq h_j \Leftrightarrow g_i \leq g_j$) and equality holds if and only if $h_i = \frac{1}{3}h$ and $g_i = \frac{1}{3}g$ with $i = 1, 2, 3$.

Without using Chebyshev's inequality we could proceed in this way:

$$(h_1 + h_2 + h_3)(g_1 + g_2 + g_3) \leq 3(h_1 g_1 + h_2 g_2 + h_3 g_3)$$
$$\Leftrightarrow \quad (kg_1 + kg_2 + kg_3)(g_1 + g_2 + g_3) \leq 3(kg_1^2 + kg_2^2 + kg_3^2)$$
$$\Leftrightarrow \quad 2(g_1^2 + g_2^2 + g_3^2) - 2(g_1 g_2 + g_2 g_3 + g_3 g_1) \geq 0$$
$$\Leftrightarrow \quad (g_1 - g_2)^2 + (g_2 - g_3)^2 + (g_1 - g_3)^2 \geq 0$$

The conclusion is as before.

Let $f = \frac{1}{3}$. Let M be the midpoint of the base corresponding to g_1 in the smaller triangle. Since $g_2 = g_3$ we also have that M is the midpoint of the base corresponding to g in the given triangle. A multiplication by a factor three from the point M carries the smaller triangle into the given triangle (since the two triangles are similar and base g_1 carries to the base g). Hence PM is a median in the given triangle and since P is carried into a vertex in the given triangle by the multiplication of three from M, we conclude that P is the centroid of the given triangle.

Alternative: It is possible to arrive at the inequality just by considering ratios: Let x_1, x_2, x_3, and x be the lengths of parallel sides of the three small triangles and the given triangle. Then it is easy to see that $x = x_1 + x_2 + x_3$. And because the triangles are all similar, the ratio of the area of a small triangle to the area of given triangle is $(x_i/x)^2$, for $i = 1, 2, 3$. So

$$f = \left(\frac{x_1}{x}\right)^2 + \left(\frac{x_2}{x}\right)^2 + \left(\frac{x_3}{x}\right)^2 = \frac{x_1^2 + x_2^2 + x_3^2}{(x_1 + x_2 + x_3)^2} \geq \frac{1}{3}$$

where the inequality follows from the relation between the regular and the square mean, or the Cauchy-Schwarz inequality, with equality when $x_1 = x_2 = x_3$.

Solution 2

The answer is $a = 2$.

We denote the polynomial $x^2 + x + a$ with $P_a(x)$, the polynomial forming the other factor of the left side with $Q(x)$ and the polynomial of the right side with $R(x)$. The polynomials are integer valued for every integer x. For $x = 0$ we get $P_a(0) = a$ and $R(0) = -90$, so a is a divisor of $90 = 2 \cdot 3 \cdot 3 \cdot 5$. For $x = -1$ we get $P_a(-1) = -184$, so a is also a divisor of $184 = 2 \cdot 2 \cdot 2 \cdot 23$. The greatest common divisor is 2, so the only possibilities for a are ± 2 and ± 1.

If $a = 1$ we get for $x = 1$ that $P_1(1) = 3$, while $R(1) = 4 - 180 = -176$ which cannot be divided by 3. If $a = -2$ we get for $x = 1$ that $P_2(1) = 0$, i.e. the left side is equal to 0, while the right side is equal to $R(1) = -176$ which is different from 0. Neither $a = 1$ nor $a = -2$ will thus work. It remains to check $a = 2$ and $a = -1$.

If $a = -1$ we get for $x = 2$ that $P_1(2) = 5$, so this possiblity can be eliminated by showing that 5 does not divide $R(2)$. As in the previous cases this can be done by evaluating $R(2)$. To that end we observe that $x^4 + 1$ is a divisor of $R(x)$, since the right side may be written as $(x^4 + 1)(x^{13} + x - 90)$. For $x = 2$ we get $x^4 + 1 = 17$ and $x^{13} + x - 90 = 8104$, neither of which is divisible by 5.

Alternatively, by Fermat's theorem $2^4 \equiv 1 \pmod 5$, so

$$R(2) = (2^4)^4 \cdot 2 + (2^4)^3 \cdot 2 + 2^4 \cdot 2 + 2 - 90(2^4 + 1) \equiv 2 + 2 + 2 + 2 \equiv 3 \pmod 5$$

so $P_1(2) = 5$ does not divide $R(2)$.

Now, the only remaining possibility is that $a = 2$, i.e. $x^2 + x + 2$ is a divisor of $R(x)$.

Remark: It can be shown that $Q(x) = (x^4 + 1)(x^{11} - x^{10} - x^9 + 3x^8 - x^7 - 5x^6 + 7x^5 + 3x^4 - 17x^3 + 11x^2 + 23x - 45)$.

Alternative: Assuming $Q(x) = x^{15} - \ldots + dx^2 + cx + b$, matching the coffecients of the lowest terms in the polynomial equation yields $ab = -90$ and $ac + b = 1$. This implies that a is a divisor of 90, and the possible values can then be further eliminated by checking which divisors yield integer solutions for c in the the second equation. This reduces the possible values of a to $1, 2, 10, -1, -2, -9, -90$. We can then eliminate $10, -2, -9, -90$ by considering the coefficient equation $ad + c = 0$ for the x^2 terms. This leaves -1, 1, and 2 as the only possiblities, but to eliminate $a = \pm 1$ with this method requires us to go through all the coefficient equations, because dividing by 1 or -1 always yields an integer solution and we will therefore not get a contradiction until in the end.

Solution 3

The answer is no. First notice that in each move two integers will be replaced with two greater integers (except in the case when the number 1 is wiped out). Notice also that from the start there are three odd integers. If one chooses to replace two odd integers on the blackboard, the number of odd integers on the blackboard decreases. If one chooses to replace two integers, which are not both odd, the number of odd integers on the blackboard is unchanged. To end up in a situation, where three of the integers on the blackboard are 2009, then it is not allowed in any move to replace two odd integers. Hence the number 2009 can only be obtained as a sum $a + b$.

In the first move that gives 2009 on the blackboard, two integers a and b are chosen such that $a + b = 2009$ and either $ab > 2009$ or $ab = 2008$. In the case $ab = 2008$, one of the two chosen numbers is equal to 1, and hence 1 will no longer appear on the blackboard. In either case, the two integers $a + b = 2009$ and ab that appear in the creation of the first 2009 cannot be used anymore to create new instances of 2009. The second 2009 can only be obtained by choosing c and d such that $c + d = 2009$ and either $cd > 2009$ or $cd = 2008$, and just as before, the numbers $c + d = 2009$ and cd cannot be used in obtaining the last 2009. So after forming two instances of 2009, there are four integers on the blackboard that have become useless for obtaining the third instance. Hence the last integer 2009 cannot be obtained.

Solution 4

We begin by determining the gold medalist using classical elimination, where we organize 16 pairs and matches, then 8 matches of the winners, 4 matches of the winners in the second round, then 2 semifinal matches and finally one match, making 31 matches altogether.

Now the second best player must have at some point lost to the best player, and as there were 5 rounds in the elimination, there are 5 candidates for the silver medal. Let C_i be the candidate who lost to the gold medalist in round i. Now let C_1 and C_2 play, the winner play against C_3, and so forth. After 4 matches we know the silver medalist; assume this was C_k.

Now the third best player must have lost against the gold medalist or against C_k or both. (If the player had lost to someone else, there would be at least 3 better players). Now C_k won $k - 1$ times in the elimination rounds, the $5 - k$ players $C_{k+1}, \ldots, C_5$, and if $k > 1$ one player C_j with $j < k$. So there are either $(k - 1) + (5 - k) = 4$ or $(k - 1) + (5 - k) + 1 = 5$ candidates for the third place. At most 4 matches are again needed to determine the bronze winner.