The 21st Nordic Mathematical Contest

Solutions

Problem 1

Find *one* solution in positive integers to the equation

$$x^2 - 2x - 2007y^2 = 0\,.$$

Solution 1. The equation can be written

$$x(x-2) = 223 \cdot (3y)^2$$

Here the prime number 223 must divide x or $x-2$. In fact, for $x = 225$ we get $x(x-2) = 15^2 \cdot 223$, which is equal to $223 \cdot (3y)^2$ for $y = 5$.

Thus, $(x, y) = (225, 5)$ is a solution.

Solution 2. The equation resembles somewhat the Pell equation

$$x^2 - 2007y^2 = 1, \tag{1}$$

which calls for finding rational approximations to $\sqrt{2007}$. These can be found either by the method of continued fractions (the first few convergents are $44/1$, $45/1$, $179/4$ and $224/5$), or it is easy to compute by hand $\sqrt{2007} \approx 44.8 = 224/5$ and check that $x = 224$, $y = 5$ is a solution of (1). To find a solution to the original equation, note that if we replace $x = 224$ by $x' = 225$, we get $x'^2 - 2007y^2 = 1 + (225^2 - 224^2) = 450 = 2x'$. Therefore $(225, 5)$ is a solution.

Problem 2

A triangle, a line and three rectangles, with one side parallel to the given line are given in such a way that the rectangles completely cover the sides of the triangle. Prove that the rectangles must completely cover the interior of the triangle.

Solution. Take any point P inside the triangle and draw through P the line parallel to the given line as well as the line perpendicular to it. These lines meet the sides of the triangle in four points. Of these four, two must be in one of the three rectangles. Now if the two points are on the same line, then the whole segment between them, P included, is in the same rectangle. If the two points, say Q and R, are on perpendicular lines, the perpendicular segments RP and PQ are also in the same rectangle. So in any case, P is in one of the rectangles.

Problem 3

The number 10^{2007} is written on a blackboard. Anne and Berit play a game where the player in turn makes one of two operations:

(i) replace a number x on the blackboard by two integer numbers a and b greater than 1 such that $x = ab$;

(ii) erase one or both of two equal numbers on the blackboard.

The player who is not able to make her turn loses the game. Who will win the game if Anne begins and both players act in an optimal way?

Solution. We describe a winning strategy for Anne. Her first move is

$$10^{2007} \to 2^{2007}, 5^{2007}.$$

We want to show that Anne can act in a such way that the numbers on the blackboard after each of her moves are of the form:

$$2^{\alpha_1}, \ldots, 2^{\alpha_k}, 5^{\alpha_1}, \ldots, 5^{\alpha_k}.$$

This is the case after Anne's first move. If Berit for example replaces 2^{α_j} by 2^{β_1} and 2^{β_2}, then Anne would replace 5^{α_j} by 5^{β_1} and 5^{β_2}. If $\alpha_i = \alpha_j$ for some (i, j), $i \neq j$, and if Berit for example erases 5^{α_i} and 5^{α_j}, then Anne would erase 2^{α_i} and 2^{α_j}. If instead Berit erases 5^{α_i} only, then Anne would answer in the same way and erase 2^{α_i}. Thus for each move Berit makes, Anne can answer with a 'symmetric' move. Since the game is finite, Berit must be the first player failing to make a move. Thus Anne has a winning strategy.

Problem 4

A line through a point A intersects a circle in two points, B and C, in such a way that B lies between A and C. From the point A draw the two tangents to the circle, meeting the circle at points S and T. Let P be the intersection of the lines ST and AC. Show that $AP/PC = 2 \cdot AB/BC$.

Solution. First we show that if we fix the points A, B, and C but vary the circle, then the point P stays fixed. To that end, suppose we have two different circles through B and C. Draw the tangents from A to one circle, meeting the circle at points S_1 and T_1, and the tangents to the other circle, meeting that circle at points S_2 and T_2. Then, according to *the power of a point theorem* (which is *the intersecting secants theorem* if, as in this case, A is a point outside the circles; the point A is said to have the same *power* with respect to the both circles):

$$AS_1^2 = AT_1^2 = AB \cdot AC = AS^2 = AT^2.$$

This implies that all the tangent points S_1, T_1, S_2, and T_2 lie on the same circle with center A. Let Q be the intersection of S_1T_1 and S_2T_2. Then by applying again the theorem of a power of a point but now with respect to the circle with center A, we have that $QS_1 \cdot QT_1 = QS_2 \cdot QT_2$ (which is *the intersecting chords theorem*). But this in turn means that the point Q has the same power with respect to the two circles we started with, and hence lies on *the radical axis* of those two circles, that is, the line BC (the radical axis is the locus of points of equal power with respect to two given circles). So Q is the intersection of AC and both S_1T_1 and S_2T_2, wich proves that the intersection point defined in the problem is the same for both circles.

Since the location of P is independent of the circle through B and C we can, without loss of generality, choose the circle with BC as diameter. Let O be the center of this circle, R its radius, $d = AO$, and $r = PO$. Then the triangles ASO and SPO are isomorphic, so $OS/AO = PO/OS$, that is, $R/d = r/R$, or $R^2 = dr$. Then finally we have

$$\frac{AP}{PC} = \frac{d-r}{R+r} = \frac{d^2-dr}{dR+dr} = \frac{d^2-R^2}{dR+R^2} = \frac{d-R}{R} = 2 \cdot \frac{d-R}{2R} = 2 \cdot \frac{AB}{BC}.$$