(1) For $1 \le k \le 4$ we have

$$\begin{aligned} 2000 - k &= f(2005-k) &&= f(f(2010-k)) \\ &= f(1999-k) &&= f(f(2004-k)) \\ &= f(1993-k). \end{aligned}$$

In particular we have $1999 = f(1998) = f(1992) = f(2004)$. Now $1995 = f(2000) = f(f(2005)) = f(1994)$. Also $f(1993) = f(f(2004)) = f(1999) = f(f(2010)) = f(2005) = 2000$ so in fact $2000 - k = f(1999-k)$ for $k = 0, 1, \ldots, 5$, and $2000 - k = f(1993-k)$ for $k = 0, 1, \ldots 4$. We claim that $f(6n+1-k) = 2000 - k$ for $n \le 333$ and $0 \le k \le 5$. By what has been said, this is true for $n = 333$, $n = 332$. Assuming it is true for $n = m+2$ we get $f(6m+1-k)) = f(f(6m+12-k)) = f(f(6(m+2)+1-(k+1)) = f(1999-k) = 2000-k$. So $f(n) = 1999$ is true for every $n = 6m \le 2004$, that is all $n \in \{0, 6, 12, \ldots, 2004\}$.

(2) It is easy to give an example of such a heptagon with two angles of 120°. We now show that a third angle of that size cannot occur.

Consider af cylic heptagon $ABCDEFG$. If the angles at two neighbouring vertices, say, B and C, are equal, then the sides AB and CD are symmetric with respect to the perpendicular bisector of segments BC. This collides with the condition that no sides have equal lengths.

All that remains is to exclude the virtual possibility of $\angle B = \angle D = \angle F = 120°$ (up to a shift of labelling). Assume this is the case. Let O be the circumcenter. Then the concave angle COA is equal to 240°, giving the convex angle COA equal to 120°. Analogously, $\angle COE = 120°$ and $\angle EOG = 120°$. But that leaves no room for the angle GOA! Contradiction ends the proof.

(3) If $\gcd(a, b) = d > 1$ then starting at the origin only lattice points (x, y) where both x and y are divisible by d can be reached so a necessary condition is

$$\gcd(a, b) = 1. \tag{1}$$

Also note that if $a+b$ is even, then starting at the origin only lattice points (x, y) with $x+y$ even can be reached (that is, coloring the lattice points black and white like on a regular chess board, either only black lattice points can be reached or only white.) Hence another necessary condition is

$$a + b \equiv 1 \pmod{2}. \tag{2}$$

We now show that conditions (1) and (2) together are sufficient:

We may assume $a, b \geq 1$ since the only pairs (a, b) with $ab = 0$ satisfying (1) and (2) are $(1,0)$ and $(0,1)$. By (1) there are positive integers r and s such that $ra - sb = 1$ or $sb - ra = 1$, say $ra - sb = 1$. By this we can construct an allowed (a, b)-knight-walk from (x, y) as follows

$$(x, y) + r(a, b) + r(a, -b) = (x + 2ra, y)$$

to the lattice point $(x + 2ra, y)$ where each step is represented by a "+" in the above display ($2r$ steps alltogether.) From this lattice point we construct likewise an allowed (a, b)-knight-walk

$$(x + 2ra, y) + s(-b, a) + s(-b, -a) = (x + 2ra - 2sb, y)$$

to the point $(x + 2ra - 2sb, y) = (x + 2, y)$. In the same way we can use allowed (a, b)-knight-walks to get from (x, y) to any of the four lattice points $(x \pm 2, y)$ or $(x, y \pm 2)$. Hence starting at the origin, all lattice points with both coordinates even can be reached.

By (2) one of the numbers a and b is even the other odd, say $a = 2\alpha + 1$ and $b = 2\beta$. Hence an allowed (a, b)-knight-walk can take us from (x, y) to

$$(x, y) + (a, b) + (-2\alpha, -2\beta) = (x + 1, y)$$

and in the same way we can get from (x, y) to any of the four lattice points $(x \pm 1, y)$ or $(x, y \pm 1)$. Hence any lattice point can be reached by an (a, b)-knight-walk in this case.

(4) The inequality can clearly be rewritten as follows:

$$\frac{1}{1+a_1}+\cdots+\frac{1}{1+a_n}\le\frac{n\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)}{n+\frac{1}{a_1}+\cdots+\frac{1}{a_n}}.$$

Dividing both the numerator and denominator of the right hand side by $\frac{1}{a_1}+\cdots+\frac{1}{a_n}$ we get that the inequality is equivalent to

$$\frac{1}{\frac{1}{a_1^{-1}}+1}+\cdots+\frac{1}{\frac{1}{a_n^{-1}}+1}\le\frac{n}{\frac{1}{\left(\frac{a_1^{-1}+\cdots+a_n^{-1}}{n}\right)}+1}. \tag{1}$$

Consider now the function $f(x)=\frac{1}{\frac{1}{x}+1}=1-\frac{1}{1+x}$ from the set of positive real numbers to itself. We will show that f is strictly concave by showing that $g(x)=\frac{1}{1+x}$ is strictly convex: For distinct positive real numbers x and y and a real $t\in]0;1[$ we have that the inequality $g(tx+(1-t)y)<tg(x)+(1-t)g(y)$ is equivalent to $t^2(y-x)^2<(y-x)^2$ which is valid since $t\in]0;1[$. Hence $f(x)$ is a strictly concave function which therefore satisfies

$$\frac{f(x_1)+\cdots+f(x_n)}{n}\le f\left(\frac{x_1+\cdots x_n}{n}\right) \tag{2}$$

with equality if and only if $x_1=\cdots=x_n$ (a special case of Jensens inequality!) Putting each $x_i=a_i^{-1}$ into (2) gives us (1) which proves the inequality, and equality holds if and only if $a_1=\cdots=a_n$.

REMARK: Knowing that

$$tf(x)+(1-t)f(y)\le f(tx+(1-t)y) \tag{3}$$

for all positive real numbers x and y and $t\in[0;1]$ with equality if and only if $x=y$ or $t\in\{0,1\}$, we can prove (2) by induction on n:

For $n = 1$ there is nothing to prove since both sides of (2) are equal to $f(x_1)$. Assuming validity for n we get that

$$\begin{aligned}
\frac{f(x_1)+\cdots+f(x_{n+1})}{n+1} &= \frac{n\left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right)+f(x_{n+1})}{n+1} \\
&\leq \frac{nf\left(\frac{x_1+\cdots x_n}{n}\right)+f(x_{n+1})}{n+1} \\
&= \frac{n}{n+1} f\left(\frac{x_1+\cdots x_n}{n}\right) + \frac{1}{n+1} f(x_{n+1})
\end{aligned}$$

with equality throughout if and only if $x_1 = \cdots = x_n$. Now putting $t = \frac{n}{n+1}$, $x = \frac{x_1+\cdots+x_n}{n}$ and $y = x_{n+1}$ into (3) we get (2) for $n+1$, where equality holds throughout if and only if $x_1 = \cdots = x_n$ and $\frac{x_1+\cdots+x_n}{n} = x_{n+1}$, that is $x_1 = \cdots = x_n = x_{n+1}$.