12th Nordic Mathematical Contest

Thursday April 2nd, 1998

Solutions

Problem 1: By entering $x = y = 0$, we find that $2f(0) = 4f(0)$, hence, $f(0) = 0$. Next, by entering $y = nx$ for a natural number n, we get

$$f((n+1)x) = 2f(x) + 2f(nx) - f((n-1)x).$$

Finding $f(nx)$ for $n = 1, 2, \ldots$ we see that $f(nx) = n^2 f(x)$. To prove this, use induction. It clearly holds for $n = 1$. By using $f(kx) = k^2 f(x)$ for $k \leq n$, we get

$$\begin{aligned} f((n+1)x) &= 2f(x) + 2f(nx) - f((n-1)x) \\ &= (2 + 2n^2 - (n-1)^2)f(x) \\ &= (n+1)^2 f(x). \end{aligned}$$

Hence, $f(nx) = n^2 f(x)$. For $x = 1/q$, we get $f(1) = f(qx) = q^2 f(x)$, so $f(1/q) = f(1)/q^2$. This makes $f(p/q) = p^2 f(1/q) = (p/q)^2 f(1)$, hence, $f(x) = ax^2$ for some rational number a.

Finally, if $f(x) = ax^2$, $f(x+y) + f(x-y) = a(x+y)^2 + a(x-y)^2 = 2ax^2 + 2ay^2 = 2f(x) + 2f(y)$, so $f(x) = ax^2$ is a solution.

Problem 2: We know that $PA \cdot PB = PC \cdot PD = PE \cdot PF$. Furthermore, M_1P being perpendicular to the corde CD, P must be the midpoint of CD: hence, $PC = PD$. Similarly, we get $PE = PF$. Thus, we find $PC = PD = PE = PF = \sqrt{PA \cdot PB}$.

Now, as C, D, E, and F all lie on a circle with centre in P, triangles CDE etc. must have a straight angle one side being a diagonal. With all angles of $CDEF$ straight, it is a rectangle.

Problem 3: a) Assume that $x_1, \ldots, x_n$ is such a sequence. Then, $x_1 + \cdots + x_n = n(n+1)/2$. This sum should be divisible by n, which is only the case if n is odd, $(n+1)/2$ being integral: $n = 2m$ makes $n(n+1)/2 = m(2m+1) = 2m^2 + m \equiv m \bmod 2m$.

With $n = 2m+1 > 1$, we demand that $n - 1 = 2m$ divide $x_1 + \cdots + x_{n-1}$. However,

$$\begin{aligned} x_1 + \cdots + x_{n-1} &= (m+1)(2m+1) - x_{n-1} \\ &\equiv m + 1 - x_{n-1} \bmod 2m, \end{aligned}$$

so $1 \leq x_{n-1} \leq n$ makes $x_n = m + 1$.

Next, demand that $n - 2 = 2m - 1$ divide $x_1 + \cdots + x_{n-2}$. Hence, as

$$\begin{aligned} x_1 + \cdots + x_{n-2} &= (m+1)(2m+1) - x_n - x_{n-1} \\ &\equiv m + 1 - x_{n-1} \bmod 2m - 1, \end{aligned}$$

we must also have $x_{n-1} \equiv m+1 \bmod 2m-1$. If $n > 3$, we get $m+1-(2m-1) < 1$ and $m+1+(2m-1) > 2m+1 = n$, so $x_{n-1} = m+1$ is the only possible positive value not exceeding n. However, this makes $x_n = x_{n-1}$, which contradicted our initial assumption.

What remains is the cases $n = 1, 3$. For $n = 1$, $x_1 = 1$. For $n = 3$, $x_3 = 2$ and $x_1, x_2 = 1$ and 3 (in any order) gives a solution. Hence, $n = 1, 3$ are the only possibilities.

b) Let $x_1 = 1$. Let us then proceed recursively.

Assume that $x_1, \ldots, x_{n-1}$ have been chosen, their sum being A. Let m be the smallest positive integer not yet used. If we set $x_{n+1} = m$, we have two restrictions on x_n:

$$A + x_n \equiv 0 \bmod n \text{ and } A + x_n + m \equiv 0 \bmod n + 1.$$

With n and $n+1$ relatively prime, the Chinese reminder theorem provides us with a solution in terms of x_n mod $n(n+1)$. By adding a sufficiently high multiple of $n(n+1)$, we can always find such a number which has not yet been used. This extends the sequence with two more x's.

Problem 4: Drawing the Pascal triangle modulo 2,

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 0 & & 1 & & & & \\ & & & 1 & & 1 & & 1 & & 1 & & & \\ & & 1 & & 0 & & 0 & & 0 & & 1 & & \\ & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & \\ 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1, \end{array}$$

one may observe that row 1 contains two copies of row 0, rows 2 to 3 contain two copies of rows 0 to 1, etc. In order to prove this, it suffices to prove that $\binom{2^k}{i} \equiv 1$ only if $i = 0$ or 2^k; using $\binom{n+1}{p} = \binom{n}{p-1} + \binom{n}{p}$, the triangles below $\binom{2^k}{0}$ and $\binom{2^k}{2^k}$ do not meet until at row 2^{k+1}, hence, are copies of rows 0 to $2^k - 1$. Alternatively, one may use

$$\binom{2^k + n}{p} = \sum_{i+j=p} \binom{2^k}{i} \binom{n}{j} \equiv \binom{2^k}{0} \binom{n}{p} + \binom{2^k}{2^k} \binom{n}{p - 2^k}.$$

If N_n is the count, $n = 2^k + m$, $m < 2^k$, we then get $N_n = 2N_m$. As $N_0 = 1$, this makes N_n a power of two, the power being the number of 1's in the binary expansion of n.

Proving that $\binom{2^k}{p} \equiv 1$ only for $p = 0, 2^k$ may be done in several ways. For one, one may use that $\binom{2^k-1}{p} \equiv 1$ for all p (this would follow from the above induction), which determines $\binom{2^k}{p}$. Another method is by counting the factors of 2. The factor $2^k!$ is divisible by 2 a total of $2^k - 1$ times. The $p!$ may be divided by 2 a total of $\lfloor p/2 \rfloor + \lfloor p/4 \rfloor + \cdots$ times and similarly for $(2^k - p)!$; $2^k/2^i \geq \lfloor p/2^i \rfloor + \lfloor (2^k - p)/2^i \rfloor$ with equality only if there is no rounding down, so the total number of factors of 2 occuring in numerator and denominator of $\binom{2^k}{p}$ is zero only if there is no rounding down in the previous. This happens only if $p = 0$ or 2^k.

Alternative solution: Let $A(n) = k$ where $2^k | n!$ but $2^{k+1} \not| n!$. As above, $A(n) = \sum_{i=1}^{\infty} \lfloor n/2^i \rfloor$. If $n = \sum_{j=0}^{k} a_j 2^j$, $a_i \in \{0, 1\}$, then we find that

$$A(n) = \sum_{1 \leq i \leq j \leq k} a_j 2^{j-i} = \sum_{j=0}^{k} a_j (2^j - 1) = n - T(n)$$

where $T(n) = \sum_{j=0}^{k} a_j$ is the sum of the binary digits.

Now, we see that $\binom{p+q}{p}$ is odd if and only if $A(p+q) = A(p) + A(q)$ which is equivalent to $T(p+q) = T(p) + T(q)$. This certainly occures if $p = \sum_{j=0}^{k} p_j 2^j$, $q = \sum_{j=0}^{k} q_j 2^j$ where $p_j, q_j \in \{0, 1\}$, $p_j + q_j < 2$, as this makes the binary representation for $p + q = \sum_{j=0}^{k} (p_j + q_j) 2^j$. We must then prove that otherwise, $T(p+q) < T(p) + T(q)$. It suffices to prove this for $q = 2^l$ and then use this result for each $q_l = 1$.

Let p be as above, $q = 2^l$. If $l > k$ or $p_l = 0$, we get $T(p+q) = T(p) + 1 = T(p) + T(q)$. Otherwise, we get $p + q = (p - 2^l) + p^{l+1}$. As $T(p - 2^l) = T(p) - 1$, induction yields $T(p+q) \leq T(p) < T(p) + T(q)$.

Now, given $p + q = n$, $n = \sum_{j=0}^{k} n_j 2^j$ fixed, we get $\binom{n}{p,q}$ odd exactly in the cases where p and q are made from splitting the 2^j for which $n_j = 1$ into two groups. As there are $T(n)$ such terms, this can be done in exactly $2^{T(n)}$ different ways.