10th Nordic Mathematical Contest

Thursday, March 11, 1996

Solutions

1. The sum of digits in 1996 is 25, and the sum of digits in $2 \cdot 1996 = 3992$ is 23. Hence, the number obtained by concatenating 78 1996s and 2 3992s has the required properties.

2. Since, for $x = 0$, the expression is not defined for any integer n, we have $x \neq 0$. Then, for any x, $x^0 + x^{-0} = 2$ is an integer. Since $x^{-n} + x^{-(-n)} = x^n + x^{-n}$, the problem is thus reduced to determining for which x the expression is an integer for any positive integer n.

   Now, due to

$$x^n + x^{-n} = (x^1 + x^{-1})(x^{n-1} + x^{-(n-1)}) - (x^{n-2} + x^{-(n-2)}),$$

   if $x^1 + x^{-1}$ is an integer, $x^n + x^{-n}$ is an integer for any integer $n \geq 1$. Hence, the condition to be satisfied is

$$x^1 + x^{-1} = m,$$

   where m is an integer. This is equvalent to a quadratic equation, whose solutions

$$x = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - 1}$$

   are real for $m \neq -1, 0, 1$.

3. Let F be the foot of the altitude from A in triangle ABC, and G the foot of the altitude from A in triangle ADE. Let, furthermore, P be the circumcentre of triangle ABC, and Q the centre of circle ADE. Without loss of generality, we assume that $\angle ACB$ is acute. Then, by periferal angles in circle ADE, $\angle EDA = \angle EFA = \angle ECF = \angle ACB$. Hence, triangles ABC and AED are similar. Now, the lines $AB = AD$ and $AC = AE$ are images of each other by reflection in the bisector of angle CAB. By this reflection, the lines AF and AG are then also images of each other, and similarly the lines AP and AQ. But then, since $AQ = AF$, $AP = AG$.

4. (a) This follows by repeted use of $f(n+a) = \dfrac{f(n)-1}{f(n)+1}$:

$$f(n+2a) = f((n+a)+a) = \frac{\dfrac{f(n)-1}{f(n)+1} - 1}{\dfrac{f(n)-1}{f(n)+1} + 1} = -\frac{1}{f(n)},$$

$$f(n+4a) = f((n+2a)+2a) = -\frac{1}{-\dfrac{1}{f(n)}} = f(n).$$

(b) If $a = 1$, we have $f(1) = f(a) = f(1995) = f(3 + 498 \cdot 4a) = f(3) = f(1 + 2a) = -\frac{1}{f(1)}$, which is impossible because $f(1)$ and $\frac{1}{f(1)}$ have the same sign. Thus, $a \neq 1$.

If $a = 2$, we have $f(2) = f(a) = f(1995) = f(3 + 249 \cdot 4a) = f(3) = f(a + 1) = f(1996) = f(4 + 249 \cdot 4a) = f(4) = f(2 + a) = \frac{f(2) - 1}{f(2) + 1}$, yielding a quadratic equation in $f(2)$ with no real solution. Thus, $a \neq 2$.

If, on the other hand, $a = 3$, f may be constructed by choosing $f(1)$, $f(2)$ and $f(3)$ arbitrarily, different from $-1, 0, 1$, and calculating the remaining values recursively by means of $f(n + 3) = \frac{f(n) - 1}{f(n) + 1}$. For $n \in \{1, 2, 3\}$, since $f(n) \neq -1, 0, 1$, none of $f(n)$, $f(n + 3) = \frac{f(n) - 1}{f(n) + 1}$, $f(n + 6) = -\frac{1}{f(n)}$, and $f(n + 9) = -\frac{f(n) + 1}{f(n) - 1}$, is equal to -1, and then, by the result in (a), $f(n) \neq -1$ for all positive integer n. Therefore, division by zero does not occur.

The function thus constructed has the required properties. In fact, by construction,

$$f(n + a) = f(n + 3) = \frac{f(n) - 1}{f(n) + 1},$$

as required. Then, by the result in (a),

$$f(n + 12) = f(n + 4a) = f(n),$$

whence,

$$f(a) = f(3) = f(3 + 166 \cdot 12) = f(1995),$$
$$f(a + 1) = f(4) = f(4 + 166 \cdot 12) = f(1996),$$
$$f(a + 2) = f(5) = f(5 + 166 \cdot 12) = f(1997),$$

as required.

$a = 3$ is thus the smallest possible value.