20^{th} Nordic Mathematical Contest

Thursday March 30, 2006

English version

Time allowed: 4 hours. Each problem is worth 5 points.

Problem 1. Let B and C be points on two fixed rays emanating from a point A such that $AB + AC$ is constant.
Prove that there exists a point $D \neq A$ such that the circumcircles of the triangels ABC pass through D for every choice of B and C.

Problem 2. The real numbers x, y and z are not all equal and fulfill

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k$$

Determine all possible values of k.

Problem 3. A sequence of positive integers $\{a_n\}$ is given by

$$a_0 = m \qquad \text{and} \qquad a_{n+1} = a_n^5 + 487 \quad \text{for all} \quad n \geq 0$$

Determine all values of m for which the sequence contains as many square numbers as possible.

Problem 4. The squares of a 100×100 chessboard are painted with 100 different colours. Each square has only one colour and every colour is used exactly 100 times.
Show that there exists a row or a column on the chessboard in which at least 10 colours are used.

Only writing and drawing sets are allowed

Solution 1. Let B and B_1 be points on one of the rays emanating from A and C and C_1 be points on the other ray emanating from A. We have to prove that the circumcircles of the triangles ABC and AB_1C_1 pass the same point D regardless of the choice of B_1 and C_1 if the condition $AB+AC = AB_1+AC_1$ is fulfilled. To determine D we make a special choice of B_1 and C_1. Let B_1 and C_1 be the reflection points of C and B, respectively, by reflection in the bisector of $\angle BAC$. The circumcircle of triangle AB_1C_1 is the reflected circumcircle of the triangle ABC and hence D must be the intersection point ($\neq A$) of the circumcircle of triangle ABC and the bisector of $\angle BAC$.
Let B_1 and C_1 be another choice. We may assume B_1 is on the line segment AB. Then C is on the line segment AC_1. From the condition $AB + AC = AB_1 + AC_1$ we get $CC_1 = BB_1$. Since $ABCD$ is a quadrilateral inscribed in a circle and AD bisects $\angle BAC$, then $BD = DC$ and $\angle C_1CD = \angle B_1BD$. So the triangles B_1BD and C_1CD are congruent. But then $\angle DB_1B = \angle DC_1C$. From this we conclude that the quadrilateral AB_1DC_1 is inscribed in a circle. So D is on the circumcircle of triangle AB_1C_1.

Solution 2. From $x + \frac{1}{y} = k$ we get $\frac{1}{x} = \frac{y}{ky-1}$. Further from $y + \frac{1}{z} = k$ we get $z = \frac{1}{k-y}$. By putting these expressions in the equation $z + \frac{1}{x} = k$ we get

$$\begin{aligned}
\frac{1}{k-y} + \frac{y}{ky-1} = k \quad &\Leftrightarrow \\
ky - 1 + y(k-y) = k(k-y)(ky-1) \quad &\Leftrightarrow \\
k^3y - k^2 - k^2y^2 + 1 - ky + y^2 = 0 \quad &\Leftrightarrow \\
ky(k^2-1) - (k^2-1) - y^2(k^2-1) = 0 \quad &\Leftrightarrow \\
(k^2-1)(ky - 1 - y^2) = 0 \quad &\Leftrightarrow \\
k = \pm 1 \vee ky - 1 - y^2 = 0 \quad &\Leftrightarrow \\
k = \pm 1 \vee k = y + \frac{1}{y} &
\end{aligned}$$

If we combine $k = y + \frac{1}{y}$ with the given equations we get $x = y = z$ and that is against the assumption. Hence $k = \pm 1$.
These values of k are possible. Example: $x = 2, y = -1, z = \frac{1}{2}$ shows that $k = 1$ is possible. By changing signs on these x, y and z we also change sign on k.

Solution 3. $m = 9$.
Notice that if a_n is a square number, then $a_n \equiv 0 \vee a_n \equiv 1 \mod 4$.
If $a_k \equiv 0 \mod 4$, then $a_{k+i} \equiv 3 \mod 4$ when i is an odd positive integer and $a_{k+i} \equiv 2 \mod 4$ when i is an even positive integer. Hence a_n is not a square number when the index is greater than k.
If $a_k \equiv 1 \mod 4$, then $a_{k+1} \equiv 0 \mod 4$. Hence a_n is not a square number when the index is greater than $k+1$.
This shows that the sequence at most contains two square numbers.
Suppose that the sequence contains two square numbers a_k and a_{k+1}, then $a_k = s^2$, where s is odd, and $a_{k+1} = s^{10} + 487 = t^2$. Let $t = s^5 + r$, then $t^2 = (s^5 + r)^2 = s^{10} + 2s^5 r + r^2$, hence $2s^5 r + r^2 = 487$.
If $s = 1$, then $r(2 + r) = 487$ which is impossible. If $s = 3$, then $486r + r^2 = 487$, and hence $r = 1$. If $s > 3$, the equation has no solutions. Hence $a_k = 9$.
Since $a_n > 487$ when $n > 0$, then $m = a_0 = 9$ (if $a_0 = 9$ then the above calculations indeed show that $a_1 = 9^5 + 487 = 244^2$ is a square number).

Solution 4. Let R_i and C_i be the number of colours used to colour the squares in row i and column i, respectively, where $i = 1, \ldots, 100$. We want to show that at least one of the integers $R_1 \ldots, R_{100}, C_1, \ldots, C_{100}$ is greater than or equal to 10.
Consider the sum $\sum_{i=1}^{100} R_i + \sum_{i=1}^{100} C_i$. This sum is equal to $\sum_{i=1}^{100} r_i + \sum_{i=1}^{100} c_i$, where $r_i =$ the number of rows containing the colour i and $c_i =$ the number of columns containing the colour i.
According to the A-G-inequality we have $r_i + c_i \geq 2\sqrt{r_i c_i}$. The colour i occurs not more than c_i times in each row, where it occurs, i.e it occurs not more than $r_i c_i$ times on the chessboard. Hence $r_i c_i \geq 100$.

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\begin{aligned}
\sum_{i=1}^{100} R_i + \sum_{i=1}^{100} C_i &= \sum_{i=1}^{100} r_i + \sum_{i=1}^{100} c_i = \sum_{i=1}^{100} (r_i + c_i) \\
&\geq \sum_{i=1}^{100} 2\sqrt{r_i c_i} \geq \sum_{i=1}^{100} 2\sqrt{100} = 2000
\end{aligned}
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From this we conclude that at least one of the integers $R_1 \ldots, R_{100}, C_1, \ldots, C_{100}$ is greater than or equal to 10.