19th Nordic Mathematical Contest – 2005

Proposed Solutions

1. Let

$$a = \sum_{k=0}^{n} a_k 10^k, \qquad 0 \le a_k \le 9, \text{ for } 0 \le k \le n-1, \ 1 \le a_n \le 9.$$

Set

$$f(a) = \prod_{k=0}^{n} a_k.$$

Since

$$f(a) = \frac{25}{8}a - 211 \ge 0,$$

$a \ge \frac{8}{25} \cdot 211 = \frac{1688}{25} > 66$. Also, $f(a)$ is an integer, and gcf(8, 25) = 1, so $8 \mid a$. On the other hand,

$$f(a) \le 9^{n-1}a_n \le 10^n a_n \le a.$$

So

$$\frac{25}{8}a - 211 \le a$$

or $a \le \frac{8}{17} \cdot 211 = \frac{1688}{17} < 100$. The only multiples of 8 between 66 and 100 are 72, 80, 88, and 96. Now $25 \cdot 9 - 211 = 17 = 7 \cdot 2$, $25 \cdot 10 - 211 = 39 \ne 8 \cdot 0$, $25 \cdot 11 - 211 = 64 = 8 \cdot 8$, and $25 \cdot 12 - 211 = 89 \ne 9 \cdot 6$. So 72 and 88 are the numbers asked for.

2. *1st Solution.* The inequality is equivalent to

$$2\left(a^2(a+b)(a+c) + b^2(b+c)(b+a) + c^2(c+a)(c+b)\right) \ge (a+b+c)(a+b)(b+c)(c+a).$$

The left hand side can be factored as $2(a+b+c)(a^3+b^3+c^3+abc)$. Because $a+b+c$ is positive, the inequality is equivalent to

$$2(a^3+b^3+c^3+abc) \ge (a+b)(b+c)(c+a).$$

After expanding the right hand side and subtracting $2abc$, we get the inequality

$$2(a^3+b^3+c^3) \ge (a^2b+b^2c+c^2a) + (a^2c+b^2a+c^2b),$$

still equivalent to the original one. But we now have two instances of the well-known inequality $x^3+y^3+z^3 \ge x^2y+y^2z+z^2x$ or $x^2(x-y)+y^2(y-z)+z^2(z-x) \ge 0$. [Proof: We may assume $x \ge y$, $x \ge z$. If $y \ge z$, write $z-x = z-y+y-z$ to obtain the equivalent and true inequality $(y^2-z^2)(y-z)+(x^2-z^2)(x-y) \ge 0$, if $z \ge y$, similarly write $x-y = x-z+z-y$, and get $(x^2-z^2)(x-z)+(x^2-y^2)(z-y) \ge 0$.]

2nd Solution. The original inequality is symmetric in a, b, c. So we may assume $a \geq b \geq c$. So

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

The Chebyshev inequality gives

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq \frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right).$$

The power mean inequality gives

$$\frac{a^2+b^2+c^2}{3} \geq \left(\frac{a+b+c}{3}\right)^2.$$

So

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq \frac{2}{9}(a+b+c)^2\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right).$$

To complete the proof, we have to show that

$$2(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq 9.$$

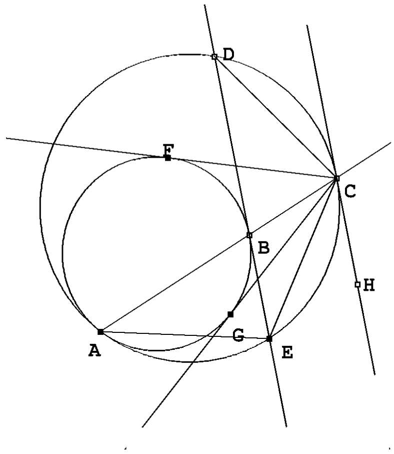
But this is equivalent to the harmonic–arithmetic mean inequality

$$\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x+y+z}{3},$$

with $x = a+b$, $y = b+c$, $z = c+a$.

3. Assume the number of girls to be g and the number of boys b. Call a position clockwise fairly strong, if, counting clockwise, the number of girls always exceeds the number of boys. No girl immediately followed by a boy has a fairly strong position. But no pair consisting of a girl and a boy following her has any effect on the fairly strongness of the other positions. So we may remove all such pairs. so we are left with at least $g - b$ girls, all in a clockwise fairly strong position. A similar count of counterclockwise fairly strong positions can be given, yielding at least $g - b$ girls in such a position. Now a sufficient condition for the existence of a girl in a strong position is that the sets consisting of the girls in clockwise and counterclockwise fairly strong position is nonempty. This is certainly true if $2(g - b) > g$, or $g > 2b$. With the numbers in the problem, this is true.

4. Draw the tangent CH to $\mathcal{C}_2$ at C. By the theorem of the angle between a tangent and chord, the angles ABH and ACH both equal the angle at A between BA and the common tangent of the circles at A. But this means that the angles ABH and ACH are equal, and $CH \| BE$. But this means that C is the midpoint of the arc DE. This again implies the equality of the angles CEB and BAE, as well as $CE = CD$. So the triangles AEC, CEB, having also a common angle ECB, are similar. So



$$\frac{CB}{CE} = \frac{CE}{AC},$$

and $CB \cdot AC = CE^2 = CD^2$. But by the power of a point theorem, $CB \cdot CA = CG^2 = CF^2$. We have in fact proved $CD = CE = CF = CG$, so the four points are indeed concyclic.