19th Nordic Mathematical Contest

April 5, 2005

Problem 1.

Find all positive integers k such that the product of the digits of k, in the decimal system, equals

$$\frac{25}{8}k - 211.$$

Problem 2.

Let a, b, and c be positive real numbers. Prove that

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq a + b + c.$$

Problem 3.

There are 2005 young people sitting around a (large!) round table. Of these at most 668 are boys. We say that a girl G is in a strong position, if, counting from G to either direction at any length, the number of girls is always strictly larger than the number of boys. (G herself is included in the count.) Prove that in any arrangement, there always is a girl in a strong position.

Problem 4.

The circle $\mathcal{C}_1$ is inside the circle $\mathcal{C}_2$, and the circles touch each other at A. A line through A intersects $\mathcal{C}_1$ also at B and $\mathcal{C}_2$ also at C. The tangent to $\mathcal{C}_1$ at B intersects $\mathcal{C}_2$ at D and E. The tangents of $\mathcal{C}_1$ passing through C touch $\mathcal{C}_1$ at F and G. Prove that D, E, F, and G are concyclic.

Time allowed: 4 hours.
Only writing and drawing instruments allowed.
Each problem is worth 5 points.