NMC 2002

Solutions

1 The arcs AD and BC are equal. Since $AD < CD$ the line PB will intersect the line DC between D and C. Also, since $AB||DC$ and $DP||AC$, we have $\angle CAB = \angle PDC$ and the arcs PC and CB are equal. Since DE is tangent to c, and AD, PC are equal, $\angle EDA = \angle ACD = \angle PBC = \angle QBC$. As $ABCD$ is inscribed in c, $\angle QCB = 180° - \angle DAB = \angle EAD$. Seeing that $ABCD$ is an isosceles trapezoid, $AD = CB$. So the triangles ADE and CBQ are congruent. But then $QC = EA$. Now $EACQ$ is a quadrilateral with a pair of opposite sides equal and parallel. Thus, $EACQ$ is a parallelogram and $EQ = AC$.

2. Let n be the original number of balls in the urn from which the ball is moved, and let a denote the sum of the numbers of these balls. Further, let m be the original number of balls in the other urn, and let b denote the sum of the numbers of the corresponding balls, i.e. $n + m = N$ and

$$a + b = 1 + 2 + \ldots + N = \frac{N(N+1)}{2}.$$

Finally, let q denote the number of the ball moved. Then

$$\frac{a-q}{n-1} = \frac{a}{n} + x$$

and

$$\frac{b+q}{m+1} = \frac{b}{m} + x,$$

from which we get

$$a = nq + xn(n-1) \tag{1}$$

and

$$b = mq - xm(m+1). \tag{2}$$

Summing (1) and (2), we get

$$N(N+1)/2 = a + b = Nq + xN(n-m-1),$$

giving

$$q = \frac{N+1}{2} - x(n-m-1) = \frac{N+1}{2} - x(N-2m-1), \tag{3}$$

i.e.

$$b = m(\frac{N+1}{2} - xn) = m(\frac{m+1}{2} + \frac{n}{2} - xn).$$

Furthermore, as $b \geq m(m+1)/2$, we have $mn(\frac{1}{2} - x) \geq 0$, or $x \leq \frac{1}{2}$.
The maximum value, $x = \frac{1}{2}$, is taken when $b = m(m+1)/2$, i.e. when the balls with the numbers $1, 2, \ldots, m$ are in the "receiving" urn, the balls with the numbers $m+1, m+2, \ldots, N$ are in the "transmitting" urn and, from (3), when $q = m+1$.

3. Since the polynomial

$$P(x) = (x + b_1)(x + b_2) \cdot \ldots \cdot (x + b_n)$$

is equal to d, say, for $x = a_1, a_2, \ldots, a_n$, the polynomial $P(x) - d$ also has the representation

$$c(x - a_1)(x - a_2) \cdot \ldots \cdot (x - a_n).$$

By identification we find that $c = 1$. Here, all $a_i\,'$s are different. For $x = -b_j$, $j = 1, 2, \ldots, n$, we get

$$\begin{aligned} P(b_j) = 0 - d &= (-b_j - a_1)(-b_j - a_2) \cdot \ldots \cdot (-b_j - a_n) \\ &= (-1)^n (a_1 + b_j)(a_2 + b_j) \cdot \ldots \cdot (a_n + b_j). \end{aligned}$$

Thus, the product $(a_1 + b_j)(a_2 + b_j) \cdot \ldots \cdot (a_n + b_j)$ is equal to $(-d)/(-1)^n = (-1)^{n+1} d$ for every j, $j = 1, 2, \ldots, n$.

4. Every selected number has the form

$$\begin{aligned} &a_0 + a_1 \cdot 10 + a_2 \cdot 100 + \ldots + a_8 \cdot 10^8 \\ &\quad = a_0 + (11 - 1)a_1 + (99 + 1)a_2 + (1001 - 1)a_3 + (9999 + 1)a_4 \\ &\quad + (100001 - 1)a_5 + (999999 + 1)a_6 + (10000001 - 1)a_7 + (99999999 + 1)a_8 \\ &\quad = (a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8) \\ &\quad + 11(a_1 + 9a_2 + 91a_3 + 909a_4 + 9091a_5 + 90909a_6 + 909091a_7 + 9090909a_8), \end{aligned}$$

i.e. every number n can be written as

$$(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8) + 11k,$$

where k is an integer. We find that

$$\begin{aligned} n &= (a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8) - 2(a_1 + a_3 + a_5 + a_7) + 11k \\ &= (1 + 2 + \ldots + 9) - 2(a_1 + a_3 + a_5 + a_7) + 11k \\ &= 44 + 1 + 11k - 2(a_1 + a_3 + a_5 + a_7). \end{aligned}$$

It follows that n is a multiple of 11 if and only if $2(a_1 + a_3 + a_5 + a_7) - 1$ is a multiple of 11, i.e. with $s = a_1 + a_3 + a_5 + a_7$, if and only if $2s = 11t + 1$ for some positive integer t. Evidently, t is an odd number. Furthermore, from

$$1 + 2 + 3 + 4 \leq s \leq 6 + 7 + 8 + 9, \text{ i.e. } 10 \leq s \leq 30,$$

we get

$$19 \leq 2s - 1 \leq 59.$$

For $t = 1$ we have $s = 6$, which is impossible.
For $t = 3$ we have $s = 17$.
For $t = 5$ we have $s = 28$.
If $t \geq 7$, then $s \geq 39$, which is impossible.
Now it remains to examine the different cases giving s=17 and s=28. For s=17 we have the following possible cases (except for permutations): $(a_2, a_4, a_6, a_8) = (1, 2, 5, 9)$, $(1, 2, 6, 8)$, $(1, 3, 4, 9)$, $(1, 3, 5, 8)$, $(1, 3, 6, 7)$, $(1, 4, 5, 7)$, $(2, 3, 4, 8)$, $(2, 3, 5, 7)$, $(2, 4, 5, 6)$. For s=28 we have the cases $(4, 7, 8, 9)$ and $(5, 6, 8, 9)$. In total we have 11 different ordered cases. But the total number of ordered 9-digit numbers is $9!/(4! \cdot 5!) = 126$, so the probability of choosing a number, which is a multiple of 11, is $11/126 < 11/121 = 1/11$. Thus, the probability is less than $1/11$, which means that Eva is correct.