50th International Mathematical Olympiad
Bremen | Germany | 2009

Language: **English**

Day: **1**

Wednesday, July 15, 2009

Problem 1. Let n be a positive integer and let $a_1, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1}-1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1-1)$.

Problem 2. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Problem 3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Thursday, July 16, 2009

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, \; f(b) \; \text{ and } \; f(b + f(a) - 1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

Time: 4 hours and 30 minutes
Each problem is worth 7 points