SOLUTIONS FOR IMO 2005 PROBLEMS

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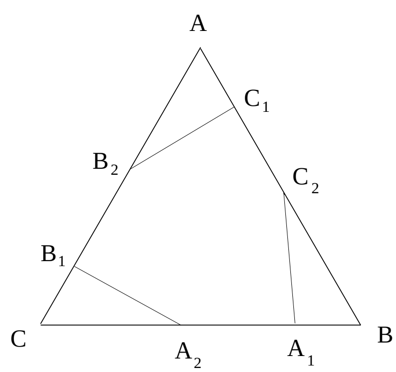
Problem 1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.



Solution: In triangles $\triangle AB_2C_1$ and $\triangle BC_2A_1$, $\angle A = \angle B$. Therefore, if $\angle BC_2A_1 > \angle AB_2C_1$, then $\angle BA_1C_2 < \angle AC_1B_2$. But $|A_1C_2| = |B_2C_1|$. Therefore, the law of sines implies

$$|BA_1| > |AC_1| \Rightarrow |BC_2| < |AB_2|.$$

On the other hand, we have $|BC_2| + |AC_1| = |AB_2| + |CB_1|$. Therefore,

$$|BC_2| < |AB_2| \Rightarrow |AC_1| > |CB_1|.$$

By a similar argument, we have:

$$\begin{aligned} |BA_1| > |AC_1| &\Rightarrow |AC_1| > |CB_1| \\ &\Rightarrow |CB_1| > |BA_1|. \end{aligned}$$

The contradiction shows that $|BA_1| = |AC_1| = |CB_1|$. Thus, the three triangles $\triangle AB_2C_1$, $\triangle BC_2A_1$ and $\triangle CA_2B_1$ are congruent. This implies that the triangle $\triangle A_2B_2C_2$ is equilateral and A_1B_2, B_1C_2 and C_1A_2 are its heights. Therefore they are concurrent.

Problem 2. Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, ..., a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

Solution: Let $A_n = \{a_1, ..., a_n\}$. Elements of A_n are distinct, because they are distinct modulo n. Observe that, for $a_i, a_j \in A_n$, $k := |a_i - a_j| < n$, because, otherwise, $a_i, a_j \in A_k$ and $a_i \equiv a_j \mod k$. Therefore,

$$\max A_n - \min A_n < n.$$

But A_n consists of n distinct integers. Therefore, for $m_n = \min A_n$,

$$A_n = \{m_n, m_n + 1, \dots, m_n + n - 1\}.$$

There are infinitely many negative and positive numbers in the sequence; therefore, all integers have to appear in our sequence. This finishes the proof.

Problem 3. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Solution: The above inequality is equivalent to

$$(1) \qquad \frac{1}{x^5 + y^2 + z^2} + \frac{1}{x^2 + y^5 + z^2} + \frac{1}{x^2 + y^2 + z^5} \leq \frac{3}{x^2 + y^2 + z^2}.$$

We have

$$\text{(Cauchy-Schwarz)} \quad (x^5 + y^2 + z^2)\,(y\,z + y^2 + z^2) \geq (\sqrt{x^5\,y\,z} + y^2 + z^2)^2$$

$$(xyz \geq 1) \qquad \geq (x^2 + y^2 + z^2)^2.$$

Therefore,

$$\frac{1}{x^5 + y^2 + z^2} \leq \frac{y\,z + y^2 + z^2}{(x^2 + y^2 + z^2)^2} \leq \frac{\frac{y^2+z^2}{2} + y^2 + z^2}{(x^2 + y^2 + z^2)^2}.$$

Similarly,

$$\frac{1}{x^2 + y^5 + z^2} \leq \frac{\frac{x^2+z^2}{2} + x^2 + z^2}{(x^2 + y^2 + z^2)^2} \quad \text{and} \quad \frac{1}{x^2 + y^2 + z^5} \leq \frac{\frac{x^2+y^2}{2} + x^2 + y^2}{(x^2 + y^2 + z^2)^2}.$$

Adding the above three inequalities proves (1).

Problem 4. Consider the sequence $a_1, a_2, \dots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \dots).$$

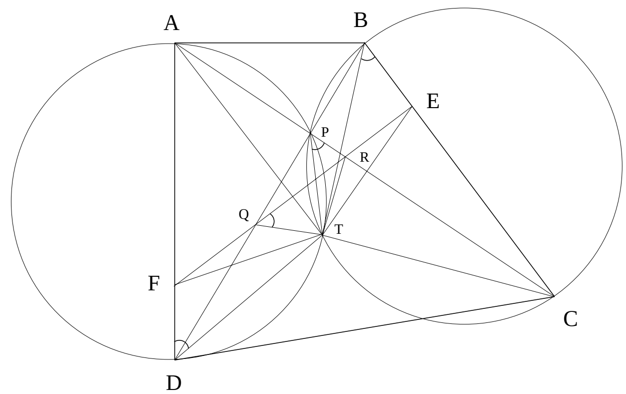
Determine all positive integers that are relatively prime to every term of the sequence.

Solution: If $p > 3$, then $2^{p-2} + 3^{p-2} + 6^{p-2} \equiv 1 \mod p$. To see this, multiply both sides by 6 to get :

$$3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} \equiv 6 \mod p,$$

which is a consequence of Fermat's little theorem. Therefore p divides a_{p-2}. Also, 2 divides a_1 and 3 divides a_2. So, there is no number other than 1 that is relatively prime to all the terms in the sequence.

Problem 5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.



Solution: The circumcircles of the triangles $\triangle PAD$ and $\triangle PBC$ intersect in points P and T. We claim that T is the desired point, i.e., P, Q, R and T lie on a circle. To prove this we show that the angles $\angle TPR$ and $\angle TQR$ are equal.

The angles $\angle ADT$ and $\angle APT$ are complimentary, therefore $\angle ADT = \angle TPC$. But $\angle TPC$ is also equal to $\angle TBC$. Therefore, $\angle ADT = \angle TBC$. Similarly, $\angle TAD = \angle TCB$. This implies that the triangles $\triangle ATD$ and $\triangle BTC$ are equal. In particular, $\triangle TFD = \triangle TEB$. This, in turn, implies that the isosceles triangles $\triangle ETF$ and $\triangle BTD$ are similar. Therefore, $\angle QFT = \angle QDT$. This means that D, Q, F and T lie on a circle, and thus $\angle FDT = \angle RQT$. But $\angle FDT$ was also equal to $\angle TPR$. Hence, $\angle TPR = \angle TQR$ which is as we claimed.

Problem 6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

Solution: Let n be the number of contestants, c be the number of contestants who solved exactly 5 problems and p_{ij} be the number of contestants who solved problems i and j, for $1 \leq i, j \leq 6$. We know that:

$$\sum_{i,j} p_{ij} \geq \binom{6}{2} \frac{2\,n+1}{5} = 6\,n + 3.$$

Also,

$$\sum_{i,j} p_{ij} \leq \binom{5}{2}\,c + \binom{4}{2}(n-c) = 6\,n + 4\,c.$$

Therefore, $4c \geq 3$. This shows that there is at least one contestant who solved exactly 5 problems. If $2\,n+1$ is not divisible by 5, then we can replace $\frac{2\,n+1}{5}$ in the above argument by $\frac{2\,n+2}{5}$ and this will imply that $4c \geq 6$ and hence there are at least two contestants who have solved 5 problems.

Now assume that $2\,n+1$ is divisible by 5, i.e., $n = 5\,k+2$, for some positive integer k. Assuming that there is exactly one person who has solved 5 problems and the rest have solved exactly 4 problems will lead to a contradiction, as we now argue in two cases. We call the only person who has solved 5 problems the champion.

Case 1. Assume that n is not divisible by 3. Let a_i be the number of contestants besides the champion who have solved problem i. Then

$$\sum_i a_i = 4\,(n-1) = 4\,n - 4.$$

Let problem 1 be the problem that the champion missed. There are 5 pairs of problems containing problem 1, and they have been solved by at least $5\frac{2\,n+1}{5} = 2\,n+1$ contestants. Since each person who has solved problem 1 has solved exactly 3 other problems, every such person has solved 3 of above 5 pairs of problems. Thus

$$3\,a_1 \geq 2\,n+1.$$

For $i > 1$, the champion has solved 4 pairs that include i. The above argument implies, $3\,a_i \geq 2\,n-3$. But, n is not divisible by 3. Therefore

$$3\,a_i \geq 2\,n-2.$$

Adding the above inequalities we get:

$$\sum 3\,a_i \geq (2\,n+1) + 5\,(2\,n-2) = 12\,n - 9,$$

which is a contradiction because the left hand side is $12\,n - 12$.

Case 2. We are left with the case where n is divisible by 3 and is of the form $5\,k+2$, i.e., $n = 15\,h-3$, and each pair of problems is solved by at least $6\,h-1$ contestants. As before, assume that the champion has not solved problem 1 and that a_1 be the number of people who have solved this problem. Each of them has solved 3 other problems. So they have each solved 3 pairs of problems containing problem 1. That is:

$$3\,a_1 \geq 5\,(6\,h-1) = 30\,h-5.$$

But a_1 is an integer; therefore,

$$a_1 \geq 10\,h-1. \tag{2}$$

Restricting our attention to 10 pairs of problems that do not contain 1, we observe that there are at least $10\,(6\,h-1)$ contestants who have solved at least one of these pairs. On the other hand, the champion has solved 10 pairs, the a_1 contestants who have solved problem 1 have solved $3\,a_1$ pairs and the rest have solved $\binom{4}{2}\,(15\,h-4-a_1)$ pairs. That is,

$$10+3\,a_1+\binom{4}{2}\,(15\,h-4-a_1) \geq 10\,(6\,h-1) \Rightarrow 3\,a_1 \leq 30\,h-4.$$

This contradicts the inequality (2). Therefore, more than one contestant solved 5 problems.