# The Georg Mohr Contest in Mathematics 2000

Thursday, January 6, 2000 at 9–13 hours

Only tools for writing and drawing are allowed

**Problem 1.** The quadrilateral $ABCD$ is a square of sidelength 1, and the points $E$, $F$, $G$ og $H$ are the midpoints of the sides.

Determine the area of quadrilateral $PQRS$.

**Problem 2.** Three identical spheres fit into a glass with rectangular sides and bottom and top in the form of regular hexagons such that every sphere touches every side of the glass.

The glass has volume 108 cm$^3$.

What is the sidelength of the bottom?

**Problem 3.** A GEORG MOHR-die is a cube on whose six sides are printed the letters G, E, O, R, M og H, respectively.

Peter has 9 totally identical GEORG MOHR-dice. Is it possible to stack them to form a tower which on each of the four sides shows the letters G E O R G M O H R in some order?

**Problem 4.** A rectangular floor is covered by a certain number of equally large quadratic tiles. The tiles along the edge are red, and the rest are white. There are equally many red and white tiles.

How many tiles can there be?

**Problem 5.** Determine all possible values of $x + \frac{1}{x}$ such that $x$ satisfies the equation

$$x^4 + 5x^3 - 4x^2 + 5x + 1 = 0,$$

and solve this equation.

*Sponsors: Georg Mohr Fonden, Dansk Matematisk Forening, Matematiklærerforeningen, UNI-C and Gyldendal.*

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Brief suggested solutions

**Problem 1.** By displacing the four small triangles as shown in the figure, a figure with the same area as quadrilateral $ABCD$ is obtained. This new figure consists of 5 equal squares. It follows that the square $PQRS$ has the area $\frac{1}{5}$.

**Problem 2.** Let the hexagon have sidelength $2a$. Then (by the figure and Pythagoras), the radius of the sphere is $a\sqrt{3}$, the cross section of the glass is $6 \cdot \frac{1}{2} \cdot a\sqrt{3} \cdot 2a$, and its heigth $3 \cdot 2 \cdot a\sqrt{3}$. For the volume, we then have $6 \cdot \frac{1}{2} \cdot a\sqrt{3} \cdot 2a \cdot 3 \cdot 2 \cdot a\sqrt{3} = 108$, whence $a = 1$. The sidelength is thus 2 cm.

**Problem 3.** The answer is no. This is proved indirectly: Say it is possible. Every letter has a "mate", namely the letter on the opposite side of the die. Each of the letters G, O and R appears twice on the front. Therefore, each of their mates appears twice on the back. Then, these mates must be G, O og R. But if, for example, G and O are mates, then R has no mate. Contradiction.

**Problem 4.** Let as far as possible every red tile be matched with a white neighbouring tile. Thus 8 red tiles are left (see figure). Then also 8 white tiles must be left. These innermost tiles make a rectangle, which must then have the dimensions $1 \times 8$ or $2 \times 4$. The entire floor then consists of either $(1 + 4) \times (8 + 4) = 60$ tiles or $(2 + 4) \times (4 + 4) = 48$ tiles.

**Problem 5.** By division by $x^2$ (obviously, $x \neq 0$) and use of $(x + \frac{1}{x})^2 = x^2 + \frac{1}{x^2} + 2$, the equation is transformed into a quadratic equation in $x + \frac{1}{x}$, which is solved:

$$x^4 + 5x^3 - 4x^2 + 5 + 1 = 0 \iff x^2 + 5x - 4 + \frac{5}{x} + \frac{1}{x^2} = 0$$
$$\iff x^2 + \frac{1}{x^2} + 2 + 5(x + \frac{1}{x}) - 6 = 0 \iff (x + \frac{1}{x})^2 + 5(x + \frac{1}{x}) - 6 = 0$$
$$\iff x + \frac{1}{x} = -6 \ \vee \ x + \frac{1}{x} = 1$$
$$\iff x^2 + 6x + 1 = 0 \ \vee \ x^2 - x + 1 = 0 \iff x = -3 \pm 2\sqrt{2}$$

(because the equation $x^2 - x + 1 = 0$ has no solution.) All possible values of $x + \frac{1}{x}$ are thus the number $-6$, and the solutions to the equation of the fourth degree are the numbers $-3 \pm 2\sqrt{2}$.