Problems for the Team Competition Baltic Way 1998

Warsaw, November 8, 1998

$4\frac{1}{2}$ hours. 5 points per problem.

1. Find all functions f of two variables, whose arguments x, y and values $f(x,y)$ are positive integers, satisfying the following conditions (for all positive integers x and y):

$$\begin{aligned} f(x,x) &= x, \\ f(x,y) &= f(y,x), \\ (x+y)f(x,y) &= yf(x,x{+}y). \end{aligned}$$

2. A triple (a,b,c) of positive integers is called *quasi-Pythagorean* if there exists a triangle with lengths of the sides a, b, c and the angle opposite to the side c equal to 120°. Prove that if (a,b,c) is a quasi-Pythagorean triple then c has a prime divisor bigger than 5.

3. Find all pairs of positive integers x, y which satisfy the equation $2x^2 + 5y^2 = 11(xy - 11)$.

4. Let P be a polynomial with integer coefficients. Suppose that for $n = 1, 2, 3, \ldots, 1998$ the number $P(n)$ is a three-digit positive integer. Prove that the polynomial P has no integer roots.

5. Let a be an odd digit and b an even digit. Prove that for every positive integer n there exists a positive integer, divisible by 2^n, whose decimal representation contains no digits other than a and b.

6. Let P be a polynomial of degree 6 and let a, b be real numbers such that $0 < a < b$. Suppose that $P(a) = P(-a)$, $P(b) = P(-b)$, $P'(0) = 0$. Prove that $P(x) = P(-x)$ for all real x.

7. Let $\mathbb{R}$ be the set of all real numbers. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying for all $x, y \in \mathbb{R}$ the equation $f(x) + f(y) = f(f(x)f(y))$.

8. Let $P_k(x) = 1 + x + x^2 + \cdots + x^{k-1}$. Show that

$$\sum_{k=1}^{n} \binom{n}{k} P_k(x) = 2^{n-1} P_n\Big(\frac{1+x}{2}\Big)$$

for every real number x and every positive integer n.

9. Let the numbers α, β satisfy $0 < \alpha < \beta < \pi/2$ and let γ and δ be the numbers defined by the conditions:

(i) $0 < \gamma < \pi/2$, and $\tan\gamma$ is the arithmetic mean of $\tan\alpha$ and $\tan\beta$;

(ii) $0 < \delta < \pi/2$, and $\dfrac{1}{\cos\delta}$ is the arithmetic mean of $\dfrac{1}{\cos\alpha}$ and $\dfrac{1}{\cos\beta}$.

Prove that $\gamma < \delta$.

10. Let $n \geq 4$ be an even integer. A regular n-gon and a regular $(n{-}1)$-gon are inscribed into the unit circle. For each vertex of the n-gon consider the distance from this vertex to the nearest vertex of the $(n{-}1)$-gon, measured along the circumference. Let S be the sum of these n distances. Prove that S depends only on n, and not on the relative position of the two polygons.

11. Let a, b, c be the lengths of the sides of a triangle. Let R denote its circumradius. Prove that

$$R \geq \frac{a^2 + b^2}{2\sqrt{2a^2 + 2b^2 - c^2}} \,.$$

When does equality hold?

12. In a triangle ABC, $\angle BAC = 90°$. Point D lies on the side BC and satisfies $\angle BDA = 2\angle BAD$. Prove that

$$\frac{1}{AD} = \frac{1}{2}\left(\frac{1}{BD} + \frac{1}{CD}\right).$$

13. In a convex pentagon $ABCDE$, the sides AE and BC are parallel and $\angle ADE = \angle BDC$. The diagonals AC and BE intersect in P. Prove that $\angle EAD = \angle BDP$ and $\angle CBD = \angle ADP$.

14. Given triangle ABC with $AB < AC$. The line passing through B and parallel to AC meets the external angle bisector of $\angle BAC$ at D. The line passing through C and parallel to AB meets this bisector at E. Point F lies on the side AC and satisfies the equality $FC = AB$. Prove that $DF = FE$.

15. Given acute triangle ABC. Point D is the foot of the perpendicular from A to BC. Point E lies on the segment AD and satisfies the equation

$$\frac{AE}{ED} = \frac{CD}{DB}.$$

Point F is the foot of the perpendicular from D to BE. Prove that $\angle AFC = 90°$.

16. Is it possible to cover a 13×13 chessboard with fourty-two pieces of dimensions 4×1 such that only the central field of the chessboard remains uncovered? (We assume that each piece covers exactly four fields of the chessboard.)

17. Let n and k be positive integers. There are nk objects (of the same size) and k boxes, each of which can hold n objects. Each object is coloured in one of k different colours. Show that the objects can be packed in the boxes so that each box holds objects of at most two colours.

18. Determine all positive integers n for which there exists a set S with the following properties:
(i) S consists of n positive integers, all smaller than 2^{n-1};
(ii) for any two distinct subsets A and B of S, the sum of the elements of A is different from the sum of the elements of B.

19. Consider a ping-pong match between two teams, each consisting of 1000 players. Each player played against each player of the other team exactly once (there are no draws in ping-pong). Prove that there exist ten players, all from the same team, such that every member of the other team has lost his game against at least one of those ten players.

20. We say that some positive integer m *covers* the number 1998, if 1, 9, 9, 8 appear in this order as digits of m. (For instance 1998 is covered by 2**1**59**9**36**98** but not by 213326798.) Let $k(n)$ be the number of positive integers that cover 1998 and have exactly n digits ($n \geq 5$), all different from 0. What is the remainder of $k(n)$ in division by 8?