“Baltic Way – 94” mathematical team contest

Hints and solutions

1. Note that

$$\begin{aligned}(x \circ y) \circ z &= x+y+z-xy-yz-xz+xyz = \\ &= (x-1)(y-1)(z-1)+1 \ .\end{aligned}$$

Hence

$$\begin{aligned}&(x \circ y) \circ z + (y \circ z) \circ x + (z \circ x) \circ y = \\ &\quad = 3\big((x-1)(y-1)(z-1)+1\big) \ .\end{aligned}$$

Now, if the required equality holds we have $(x-1)(y-1)(z-1) = -1$. There are only four possible decompositions of -1 into a product of three integers. Thus we have three such triples $(0,0,0)$, $(0,2,2)$, $(2,0,2)$ and $(2,2,0)$.

2. a) Suppose we have the opposite inequality $a_{i-1} + a_{i+1} \geqslant 2a_i$ for all $i = 2, \ldots, 8$. Let $a_k = \max\limits_{1 \leqslant i \leqslant 9} a_i$, then we have $a_{k-1} = a_{k+1} = a_k$, $a_{k-2} = a_{k-1} = a_k$, etc. Finally we get $a_1 = a_k$, a contradiction.

b) Suppose now $a_{i-1}+a_{i+1} \geqslant 1.9a_i$, i.e. $a_{i+1} \geqslant 1.9a_i - a_{i-1}$ for all $i = 2, \ldots, 8$ and let $a_k = \max\limits_{1 \leqslant i \leqslant 9} a_i$. We can multiply all numbers $a_1, \ldots, a_9$ by the same positive constant without changing the situation in any way, so we assume $a_k = 1$. Then we have $a_{k-1} + a_{k+1} \geqslant 1.9$ and hence $0.9 \leqslant a_{k-1}, a_{k+1} \leqslant 1$. Moreover, at least one of the numbers a_{k-1}, a_{k+1} must be greater or equal than 0.95 — let us assume $a_{k+1} \geqslant 0.95$. Now, we consider two subcases:

b1) $k \geqslant 5$. Then we have

$$\begin{aligned}1 \geqslant a_{k+1} &\geqslant 0.95 > 0 \ ; \\ 1 \geqslant a_{k+2} &\geqslant 1.9a_{k+1} - a_k \geqslant 1.9 \cdot 0.95 - 1 = 0.805 > 0 \ ; \\ a_{k+3} &\geqslant 1.9a_{k+2} - a_{k+1} \geqslant 1.9 \cdot 0.805 - 1 = 0.5295 > 0 \ ; \\ a_{k+4} &\geqslant 1.9a_{k+3} - a_{k+2} \geqslant 1.9 \cdot 0.5295 - 1 = 0.00605 > 0 \ .\end{aligned}$$

So in any case we have $a_9 > 0$, a contradiction.

b2) $k \leqslant 4$. In this case we obtain

$$\begin{aligned}1 \geqslant a_{k-1} &\geqslant 0.9 > 0 \ ; \\ a_{k-2} &\geqslant 1.9a_{k-1} - a_k \geqslant 1.9 \cdot 0.9 - 1 = 0.71 > 0 \ ; \\ a_{k-3} &\geqslant 1.9a_{k-2} - a_{k-1} \geqslant 1.9 \cdot 0.71 - 1 = 0.349 > 0 \ .\end{aligned}$$

and hence $a_1 > 0$, contrary to the condition of the problem.

3. The expression is well-defined only for $|x|,\ |y| \leqslant 1$ and we can assume that $x,\ y \geqslant 0$. Let $x = \cos\alpha$ and $y = \cos\beta$ for some $0 \leqslant \alpha,\ \beta \leqslant \frac{\pi}{2}$. This reduces the expression to

$$\begin{aligned}&\cos\alpha\cos\beta + \cos\alpha\sin\beta + \cos\beta\sin\alpha - \sin\alpha\sin\beta = \\ &\quad = \cos(\alpha+\beta) + \sin(\alpha+\beta) = \sqrt{2}\cdot\sin(\alpha+\beta+\frac{\pi}{4})\end{aligned}$$

which does not exceed $\sqrt{2}$. The equality holds when $\alpha+\beta+\frac{\pi}{4} = \frac{\pi}{2}$, for example when $\alpha = \frac{\pi}{4}$ and $\beta = 0$, i.e. $x = \frac{\sqrt{2}}{2}$ and $y = 1$.

Answer: the largest value of the expression is $\sqrt{2}$.

4. Inverting the relation gives

$$\begin{aligned}\frac{q}{p} &= \frac{1}{\sqrt{n+1}+\sqrt{n-1}} = \\ &= \frac{\sqrt{n+1}-\sqrt{n-1}}{(\sqrt{n+1}+\sqrt{n-1})(\sqrt{n+1}-\sqrt{n-1})} = \\ &= \frac{\sqrt{n+1}-\sqrt{n-1}}{2}\ .\end{aligned}$$

Hence we get the system of equations

$$\begin{cases}\sqrt{n+1}+\sqrt{n-1} = \dfrac{p}{q}\ , \\ \sqrt{n+1}-\sqrt{n-1} = \dfrac{2q}{p}\ .\end{cases}$$

Adding these equations and dividing by 2 gives $\sqrt{n+1} = \frac{2q^2+p^2}{2pq}$. This implies $4np^2q^2 = 4q^4 + p^4$.

Suppose now that n , p and q are all positive integers with p and q relatively prime. The relation $4np^2q^2 = 4q^4+p^4$ shows that p^4 , and hence p , is divisible by 2 . Letting $p = 2P$ we obtain $4nP^2q^2 = q^4 + 4P^4$ which shows that q must also be divisible by 2 . This contradicts the assumption that p and q are relatively prime.

Answer: there are no such positive integers n .

5. Observe first that if a and b are two different integers then $p(a) - p(b)$ is divisible by $a - b$. Suppose now that $p(a) = 1$ and $p(b) = 3$ for some integers a and b . If we have $p(c) = 2$ for some integer c then $c - b = \pm 1$ and $c - a = \pm 1$, hence there can be at most one such integer c .

Answer: No, it cannot.

6. Since the number of congruence classes modulo q is finite, there exist two non-negative integers i and j with $i > j$ which satisfy $2^i \equiv 2^j \pmod{q}$. Hence, q divides the number $2^i - 2^j = 2^j(2^{i-j} - 1)$. Since q is odd, q has to divide $2^{i-j} - 1$. Now it suffices to multiply the numerator and denominator of the fraction $\dfrac{p}{q}$ by $\dfrac{2^{i-j}-1}{q}$.

7. The sum has an even number of terms; they can be joined in pairs in such a way that the sum is the sum of the terms
$$\frac{1}{k^3} + \frac{1}{(p-k)^3} = \frac{p^3 - 3p^2k + 3pk^2}{k^3(p-k)^3} \quad .$$
The sum of all terms of this type has a denominator in which every prime factor is less than p while the numerator has p as a factor.

8. We first show this for odd numbers $a = 2i+1 \geqslant 3$. Put $c = 2k+1$ and $b = 2k$. Then $c^2 - b^2 = (2k+1)^2 - (2k)^2 = 4k + 1 = a^2$. Now $a = 2i + 1$ and thus $a^2 = 4i^2 + 4i + 1$ and $k = i^2 + i$. Furthermore, $c > b = 2i^2 + 2i > 2i + 1 = a$. Since any multiple of a Pythagorean triple (i.e. a triple of integers (x, y, z) such that $x^2 + y^2 = z^2$) is also a Pythagorean triple we see that the statement is also true for all even numbers which have an odd factor. Hence only the powers of 2 remain. But for 8 we have the triple $8, 15, 17$ and hence all higher powers of 2 are also minimum values of such a triple.

9. Considering the equality $2^a + 3^b = n^2$ modulo 3 it is easy to see that a must be even. Obviously n is odd so we may take $a = 2x$, $n = 2y+1$ and write the equality as $4^x + 3^b = (2y+1)^2 = 4y^2 + 4y + 1$. Hence $3^b \equiv 1 \pmod{4}$ which implies $b = 2z$ for some positive integer z . So we get $4^x + 9^z = (2y+1)^2$ and $4^x = (2y + 1 - 3^z)(2y + 1 + 3^z)$. Both factors on the right-hand side are even numbers but at most one of them is divisible by 4 (since their sum is not divisible by 4). Hence $2y + 1 - 3^z = 2$ and $2y + 1 + 3^z = 2^{2x-1}$. These two equalities yield $2 \cdot 3^z = 2^{2x-1} - 2$ and $3^z = 4^{x-1} - 1$. Clearly $x > 1$ and a simple argument modulo 10 gives $z = 4d + 1$, $x - 1 = 2e + 1$ for some non-negative integers d and e . Substituting, we get $3^{4d+1} = 4^{2e+1} - 1$ and $3 \cdot (80 + 1)^d = 4^{2e+1} - 1$. If $d \geqslant 1$ then $e \geqslant 1$, a contradiction (expanding the left-hand expression and moving everything to the left we find that all summands but one are divisible by 4^2). Hence $e = d = 0$ and $z = 1$, $b = 2$, $x = 2$ and $a = 4$, i.e. we obtain the classical $2^4 + 3^2 = 4^2 + 3^2 = 5^2$.
Answer: $a = 4$, $b = 2$.

10. Consider all positive integers with $2n$ digits satisfying conditions (a) and (b) of the problem. Let the number of such integers beginning with 1, 2, 3, 4 and 5 be a_n, b_n, c_n, d_n and e_n, respectively. Then, for $n = 1$ we have $a_1 = 1$ (integer 12), $b_1 = 2$ (integers 21 and 23), $c_1 = 2$ (integers 32 and 34), $d_1 = 2$ (integers 43 and 45) and $e_1 = 1$ (integer 54). Observe that $c_1 = a_1 + e_1$.

Suppose now that $n > 1$, i.e. the integers have at least four digits. If an integer begins with the digit 1 then the second digit is 2 while the third can be 1 or 3. This gives the relation

$$a_n = a_{n-1} + c_{n-1} \ . \tag{1}$$

Similarly, if the first digit is 5, then the second is 4 while the third can be 3 or 5. This implies

$$e_n = c_{n-1} + e_{n-1} \ . \tag{2}$$

If the integer begins with 23 then the third digit is 2 or 4. If the integer begins with 21 then the third digit is 2. From this we can conclude that

$$b_n = 2b_{n-1} + d_{n-1} \ . \tag{3}$$

In the same manner we can show that

$$d_n = b_{n-1} + 2d_{n-1} \ . \tag{4}$$

If the integer begins with 32 then the third digit must be 1 or 3 and if it begins with 34 the third digit is 3 or 5. Hence

$$c_n = a_{n-1} + 2c_{n-1} + e_{n-1} \ . \tag{5}$$

From (1), (2) and (5) it follows that $c_n = a_n + e_n$, which is true for all $n = 1, 2, 3, \ldots$. On the other hand, adding the relations (1) – (5) results in

$$a_n + b_n + c_n + d_n + e_n = 2a_{n-1} + 3b_{n-1} + 4c_{n-1} + 3d_{n-1} + 2e_{n-1}$$

and, since $c_{n-1} = a_{n-1} + e_{n-1}$,

$$a_n + b_n + c_n + d_n + e_n = 3(a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} + e_{n-1}) \ .$$

Thus the number of integers satisfying conditions (a) and (b) increases three times when we increase the number of digits by 2. Since the number of such integers with two digits is 8 and $1994 = 2 + 2 \cdot 996$, the number of integers satisfying all three conditions is $8 \cdot 3^{996}$.

Answer: the number of such integers is $8 \cdot 3^{996}$.

11. We have $\angle NAS = \angle NBS = 90^\circ$ (see Fig. 1). Thus, the triangles $NA'S$ and NSA are similar. Also, the triangles $B'NS$ and SNB are similar and the triangles NSA and SNB are congruent. Hence, the triangles $NA'S$ and $B'NS$ are similar which implies $\dfrac{SA'}{SN} = \dfrac{SN}{SB'}$ and $SA' \cdot SB' = SN^2$.

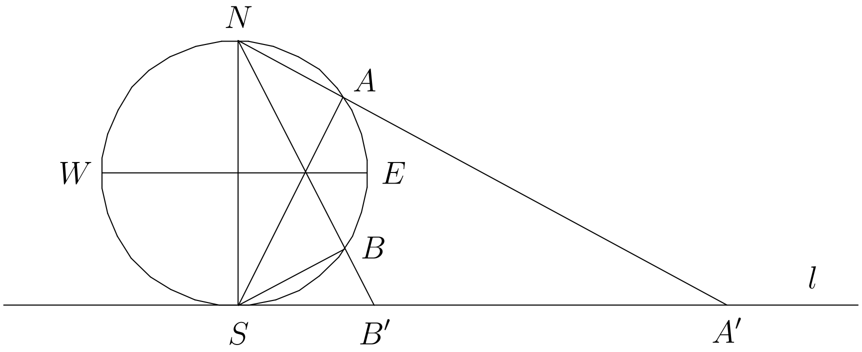

Figure 1

12. We shall prove that the lines S_1O_1 , S_2O_2 , S_3O_3 are the bisectors of the angles of the triangle $S_1S_2S_3$. Let O and r be the centre and radius of the inscribed circle C of the triangle $A_1A_2A_3$. Further, let P_1 and H_1 be the points where the inscribed circle of the triangle $A_1S_2S_3$ (with the centre O_1 and radius r_1) touches its sides A_1S_2 and S_2S_3 , respectively (see Fig. 2). To show that S_1O_1 is the bisector of the angle $\angle S_3S_1S_2$ it is sufficient to prove that O_1 lies on the circumference of circle C , for in this case the arcs O_1S_2 and O_1S_3 will obviously be equal. To prove this, first note that as $A_1S_2S_3$ is an isosceles triangle the point H_1 , as well as O_1 , lies on the straight line A_1O . Now, it suffices to show that $|OH_1| = r - r_1$. Indeed, we have

$$\frac{r-r_1}{r} = 1 - \frac{r_1}{r} = 1 - \frac{|O_1P_1|}{|OS_2|} = 1 - \frac{|P_1A_1|}{|S_2A_1|} = \frac{|S_2A_1| - |P_1A_1|}{|S_2A_1|} =$$
$$= \frac{|S_2P_1|}{|S_2A_1|} = \frac{|S_2H_1|}{|S_2A_1|} = \frac{|OH_1|}{|OS_2|} = \frac{|OH_1|}{r} .$$

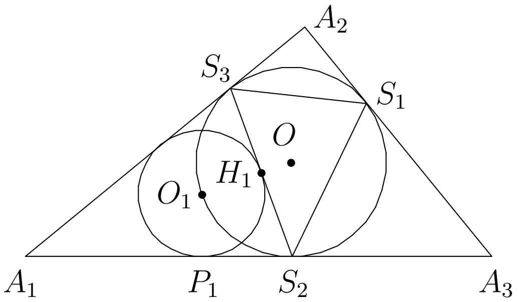

Figure 2

13. Let $PQRS$ be a square which has the property described in the problem. Clearly, $a > 2$. Let $P'Q'R'S'$ be the square inside $PQRS$ whose sides are at distance 1 from the sides of $PQRS$, and, consequently, are of length $a-2$. Since all the five disks are inside $PQRS$, their centers are inside $P'Q'R'S'$. Divide $P'Q'R'S'$ into four qongruent squares of sidelength $\frac{a}{2}-1$. By the pigeonhole principle, at least two of the five centers are in the same small square. Their distance, then, is at most $\sqrt{2}\left(\frac{a}{2}-1\right)$. Since the distance has to be at least 2, we have $a \geqslant 2+2\sqrt{2}$. On the other hand, if $a = 2+2\sqrt{2}$, we can place the five disks in such a way that one is centered at the center of $PQRS$ and the other four have centers at P' , Q' , R' , and S' .

14. Clearly, the inequality $a > b$ implies $\alpha > \beta$ and similarly $a < b$ implies $\alpha < \beta$, hence $(a-b)(\alpha-\beta) \geqslant 0$ and $a\alpha + b\beta \geqslant a\beta + b\alpha$. Dividing the last equality by $\alpha\beta$ we get

$$\frac{a}{\beta}+\frac{b}{\alpha} \geqslant \frac{a}{\alpha}+\frac{b}{\beta}\ . \tag{1}$$

Similarly we get

$$\frac{a}{\gamma}+\frac{c}{\alpha} \geqslant \frac{a}{\alpha}+\frac{c}{\gamma} \tag{2}$$

and

$$\frac{b}{\gamma}+\frac{c}{\beta} \geqslant \frac{b}{\beta}+\frac{c}{\gamma}\ . \tag{3}$$

To finish the proof it suffices to add the inequalities (1) – (3).

15. Consider a triangle ABC with all its sides and heights having integer lengths. From the cosine theorem we conclude that $\cos\angle A$, $\cos\angle B$ and $\cos\angle C$ are rational numbers. Let AH be one of the heights of the triangle ABC, with the point H lying on the straight line determined by the side BC. Then $|BH|$ and $|CH|$ must be rational and hence integer (consider the Pythagorean theorem for the triangles ABH and ACH). Now, if $|BH|$ and $|CH|$ have different parity then $|AB|$ and $|AC|$ also have different parity and $|BC|$ is odd. If $|BH|$ and $|CH|$ have the same parity then $|AB|$ and $|AC|$ also have the same parity and $|BC|$ is even. In both cases the perimeter of triangle ABC is an even number and hence cannot be equal to 1995.

    Remark. In the solution we only used the fact that all three sides and *one* height of the triangle ABC are integers.

16. It suffices to prove that if the distance between the centres of two Hedgehogs is less than 0.2 then these Hedgehogs intersect. To show this, consider two Hedgehogs with their centres at points O and M respectively such that $|OM| < 0.2$. Let A, B, C be the endpoints of the needles of the first Hedgehog (see Fig. 3) and draw a straight line l parallel to AC through the point M. As $|AC| = \sqrt{3}$ implies $|KL| \leqslant \frac{0.2}{0.5} \cdot |AC| < 1$ and the second Hedgehog has at least one of its needles pointing inside the triangle OKL, this needle intersects the first Hedgehog.

    Remark: If the Hedgehogs can move their needles so that the angles between them can take any positive value then there can be an infinite number of Hedgehogs on the Wonder Island.

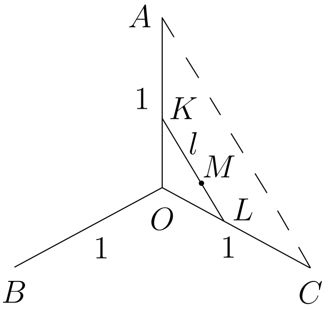

Figure 3

17. Let $a_1, \ldots, a_{13}$ be the numbers of towns on each island. Suppose there exist numbers i and j such that $a_i \geqslant a_j > 1$ and consider an arbitrary town A on the j-th island. The number of ferry connections from town A is equal to $25 - a_j$. On the other hand, if we "move" town A to the i-th island then there will be $25 - (a_i + 1)$ connections from town A while no other connections will be affected by this move. Hence, the smallest number of connections will be

achieved if there are 13 towns on one island and one town on each of the other 12 islands. In this case there will be $13 \cdot 12 + \frac{12 \cdot 11}{2} = 222$ connections.

18. Suppose we have assigned the labels in the required manner. When a point has label 1 then there can be no more occurrences of label 1 on the two lines that intersect at that point. Therefore, the number of intersection points labelled with 1 has to be exactly $\frac{n}{2}$, i.e. n must be even. Now, let n be an even number and denote the n lines by $l_1, l_2, \ldots, l_n$ First write the lines l_i in the following table:

$$\begin{array}{cccccc} & & l_3 & l_4 & \ldots & l_{n/2+1} \\ l_1 & l_2 & & & & \\ & & l_n & l_{n-1} & \ldots & l_{n/2+2} \end{array}$$

and then rotate the picture $n-1$ times:

$$\begin{array}{cccccc} & & l_2 & l_3 & \ldots & l_{n/2} \\ l_1 & l_n & & & & \\ & & l_{n-1} & l_{n-2} & \ldots & l_{n/2+1} \end{array}$$

$$\begin{array}{cccccc} & & l_n & l_2 & \ldots & l_{n/2-1} \\ l_1 & l_{n-1} & & & & \\ & & l_{n-2} & l_{n-3} & \ldots & l_{n/2} \end{array}$$

etc.

According to these tables, we can join the lines in pairs in $n-1$ different ways — l_1 with the line next to it and every other line with the line directly above or under it. Now we can assign the label i to all the intersection points of the pairs of lines shown in the i-th table.

19. We call two spies A and B *neutral* to each other if neither A watches on B nor B watches on A.

Denote the spies $A_1, A_2, \ldots, A_{16}$. Let a_i, b_i and c_i denote the number of spies that watch on A_i, the number of that are watched by A_i and the number of spies neutral to A_i, respectively. Clearly, we have

$$\begin{aligned} a_i + b_i + c_i &= 15 \\ a_i + c_i &\leqslant 8 \\ b_i + c_i &\leqslant 8 \end{aligned}$$

for any $i = 1, \ldots, 16$ (if any of the last two inequalities does not hold then there exist 10 spies who cannot be numbered in the required manner). Combining the relations above we find $c_i \leqslant 1$. Hence, for any spy, the number of his neutral collegues is 0 or 1 .

Now suppose there is a group of 11 spies that cannot be numbered as required. Let B be an arbitrary spy in this group. Number the other 10 spies as $C_1, C_2, \ldots, C_{10}$ so that C_1 watches on C_2 , ..., C_{10} watches on C_1 . Suppose there is no spy neutral to B among C_1 , ..., C_{10} . Then, if C_1 watches on B then B cannot watch on C_2 as otherwise $C_1, B, C_2, \ldots, C_{10}$ would form a 11-cycle. So C_2 watches on B , etc. As some of the spies $C_1, C_2, \ldots, C_{10}$ must watch on B we get all of them watching on B , a contradiction. Therefore, each of the 11 spies must have exactly one spy neutral to him among the other 10 — but this is impossible.

20. Consider the side AB of the big triangle ABC as “horizontal” and suppose the statement of the problem does not hold. The side AB contains 3001 vertices $A = A_0, A_1, \ldots, A_{3000} = B$ of 3 colours. Hence, there are at least 1001 vertices of one colour, e.g. red. For any two red vertices A_k and A_n there exists a unique vertex B_{kn} such that the triangle $B_{kn}A_kA_n$ is equilateral. That vertice B_{kn} cannot be red. For different pairs (k, n) the corresponding vertices B_{kn} are different, so we have at least $C_{1001}^2 > 500000$ vertices of type B_{kn} that cannot be red. As all these vertices are situated on 3000 horisontal lines, there exists a line L which contains more than 160 vertices of type B_{kn} , each of them coloured in one of the two remaining colours. Hence there exist at least 81 vertices of the same colour, e.g. blue, on line L . For every two blue vertices B_{kn} and B_{ml} on line L there exists a unique vertex C_{knml} such that:

   1) C_{knml} lies above the line L ;
   2) The triangle $C_{knml}B_{kn}B_{ml}$ is equilateral;
   3) $C_{knml} = B_{pq}$ where $p = \min(k, m)$ and $q = \max(n, l)$.

   Different pairs of vertices B_{kn} belonging to line L define different vertices C_{knml} . So we have at least $C_{81}^2 > 3200$ vertices of type C_{knml} that can be neither blue nor red. As the number of these vertices exceeds the number of horizontal lines, there must be two vertices C_{knml} and C_{pqrs} on one horizontal line. Now, these two vertices define a new vertex $D_{knmlpqrs}$ that cannot have any of the three colours, a contradiction.

   Remark: The minimal size of the big triangle that can be handled by this proof is 2557 .