“Baltic Way – 92” mathematical team contest

Vilnius, November 7, 1992

Hints and solutions

1. Since $q-p=2k$ is even, we have $p+q=2(p+k)$. It is clear that $p<p+k<p+2k=q$. Therefore $p+k$ is not prime and, consequently, is a product of two positive integers greater than 1.

2. Consider numbers of the form p^{p^n-1} where p is an arbitrary prime number and $n=1,2,\ldots$.

3. For any natural number n, we have $n^2\equiv 0$ or $n^2\equiv 1 \pmod 4$ and $n^3\equiv 0$ or $n^3\equiv \pm 1 \pmod 9$. Thus $\{\,36n+3 \mid n=1,2,\ldots\}$ is a progression with the required property.

4. There is no such sexagon. The sum of any six consecutive positive integers is odd. On the other hand, the sum of squares of lengths of the sexagon's sides is equal to the sum of squares of their projections onto the two axes. But the sum of squares of the projections has the same parity as the sum of the projections themselves, the latter being obviously even.

5. Use the identity $\big(a^2+b^2+(a+b)^2\big)^2=2\big(a^4+b^4+(a+b)^4\big)$.

6. Note that

$$\frac{k^3-1}{k^3+1}=\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}=\frac{(k-1)(k^2+k+1)}{(k+1)\big((k-1)^2+(k-1)+1\big)}\,.$$

After obvious cancellations we get

$$\prod_{k=2}^{100}\frac{k^3-1}{k^3+1}=\frac{1\cdot 2\cdot(100^2+100+1)}{100\cdot 101\cdot(1^2+1+1)}>\frac{2}{3}\,.$$

7. The first of these numbers is less than

$$\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{1992}}}}}}\right\}1992=\left.a^{a^{\cdot^{\cdot^{\cdot^{1992}}}}}\right\}1991=\ldots=1992\,.$$

8. Since 2^x must be positive, we have $\dfrac{2x+4}{4-x}>0$ yielding $-2<x<4$. Thus it suffices to check the points $-1,\ 0,\ 1,\ 2,\ 3$. The three solutions are $x=0,\,1,\,2$.

9. Consider the derivative $f'(x)=3x^2+2ax+b$. Since $b<0$, it has two real roots x_1 and x_2. Since $f(x)\to\pm\infty$ as $x\to\pm\infty$, it is sufficient to check that $f(x_1)$ and $f(x_2)$ have different signs, i.e., $f(x_1)f(x_2)<0$. Dividing $f(x)$ by $f'(x)$ and using the equality $ab=9c$ we find that the remainder is equal to $x(\frac{2}{3}b-\frac{2}{9}a^2)$. Now, as $x_1x_2=\dfrac{b}{3}<0$ we have $f(x_1)f(x_2)=x_1x_2(\frac{2}{3}b-\frac{2}{9}a^2)^2<0$.

10. Let $p(x)=ax^4+bx^3+cx^2+dx+e$ with $a\neq 0$. From (i) – (iii) we get $b=d=0$, $a>0$ and $e=1$. From (iv) it follows that $p'(x)=4ax^3+2cx$ has at least two different real roots. Since $a>0$, then $c<0$ and $p'(x)$ has three roots $x=0$, $x=\pm\sqrt{\dfrac{-c}{2a}}$. The minimum points mentioned in (iv) must be $x=\pm\sqrt{\dfrac{-c}{2a}}$, so $2\sqrt{\dfrac{-c}{2a}}=2$ and $c=-2a$. Finally, by (ii) we have $p(x)=a(x^2-1)^2+1-a\geqslant 0$ for all x, which implies $0<a\leqslant 1$. It is easy to check that every such polynomial satisfies the conditions (i) – (iv).

11. By condition (iii) we have $f(1) = 1$. Applying condition (iii) to each of (i) and (ii) gives two new conditions (i′) and (ii′) taking care of $q > 2$ and $\frac{1}{2} \leqslant q < 1$ respectively. Now, for any rational number $\frac{a}{b} \neq 1$ we can use (i), (i′), (ii) or (ii′) to express $f\left(\frac{a}{b}\right)$ in terms of $f\left(\frac{a'}{b'}\right)$ where $a' + b' < a + b$. The recursion therefore finishes in a finite number of steps, when we can use $f(1) = 1$. Thus we have established that such a function f exists, and it is uniquely defined by the given conditions.

Remark. Initially it was also required to determine all fixed points of the function f, i.e., all solutions q of the equation $f(q) = q$, but the Jury of the contest decided to simplify the problem. We present here a solution for the complete one. First note that if q is a fixed point, then so is $\frac{1}{q}$. By (i), if $0 < q < \frac{1}{2}$ is a fixed point, then $f\left(\frac{q}{1-2q}\right) = q - 1 < 0$ which is impossible, so there are no fixed points $0 < q < \frac{1}{2}$ or $q > 2$. Now, for a fixed point $1 \leqslant \frac{a}{b} \leqslant 2$ (ii) easily gives us that $\frac{a}{b} - 1 = \frac{a-b}{b}$ and $\frac{b}{a-b}$ are fixed points too. It is easy to see that $1 \leqslant \frac{b}{a-b} \leqslant 2$ (the latter holds because $\frac{b}{a-b}$ is a fixed point). As this new fixed point has the sum of its numerator and denominator strictly less than $\frac{a}{b}$ we can continue in this manner until, in a finite number of steps, we arrive to the fixed point 1. By reversing the process, any fixed point $q > 1$ can be constructed by repeatedly using the condition that if $\frac{a}{b} > 1$ is a fixed point then so is $\frac{a+b}{a}$, starting with $a = b = 1$. It is now an easy exercise to see that these fixed points have the form $\frac{F_{n+1}}{F_n}$ where $\{F_n\}_{n\in\mathbb{N}}$ is the sequence of Fibonacci numbers.

12. We show that $L = 1$ is the only possible value. Assume $L > 1$, then there exists a number N such that for any $n \geqslant N$ we have $\frac{\varphi(n)}{n} > 1$ and thus $\varphi(n) \geqslant n + 1 \geqslant N + 1$. But then φ cannot be bijective, since the numbers $1, 2, \ldots, N-1$ cannot be bijectively mapped onto $1, 2, \ldots, N$.

    Now assume $L < 1$. Since φ is bijective we clearly have $\varphi(n) \to \infty$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \frac{\varphi^{-1}(n)}{n} = \lim_{n\to\infty} \frac{\varphi^{-1}(\varphi(n))}{\varphi(n)} = \lim_{n\to\infty} \frac{n}{\varphi(n)} = \frac{1}{L} > 1 ,$$

    i.e. $\lim\limits_{n\to\infty} \frac{\varphi^{-1}(n)}{n} > 1$, which is a contradiction since φ^{-1} is also bijective.

13. Since $(x_i + y_i)^2 \geqslant 4x_iy_i$, it is sufficient to prove that

$$\Bigl(\sum_{i=1}^{n} \frac{1}{x_iy_i}\Bigr)\Bigl(\sum_{i=1}^{n} x_iy_i\Bigr) \geqslant n^2 .$$

    This can easily be done by induction using the fact that $a + \frac{1}{a} \geqslant 2$ for any $a > 0$.

14. Consider the town A from which a maximum number of towns can be reached. Suppose there is a town B which cannot be reached from A. Then A can be reached from B and so one can reach more towns from B than from A, a contradiction.

15. Start assigning the species to cages in an arbitrary order. Since for each species there are at most three species incompatible with it, we can easily add it in one of the four cages.

Remark. Initially the problem was posed as follows: "... He plans to put *two* species in each cage ...". Because of a misprint the word "two" disappeared, and the problem became actually trivial. Let us give a solution to the original problem. Start with the distribution obtained above. If in some cage A there are more than three species, then there is also a cage B with at most one species and this species is compatible with at least one species in cage A which we can then transfer to cage B. Thus we may assume that there are at most three species in each cage. If there are two cages with 3 species then we can obviously transfer one of these 6 species to one of the remaining two cages. Now, assume the four cages contain 1, 2, 2 and 3 species respectively. If the species in the first cage is compatible with one in the fourth cage then transfer that species to the first cage, and we are done. Otherwise, for an arbitrary species X in the fourth cage there exists a species compatible with it in either the second or the third cage. Transfer the other species from that cage to the first cage, and then X to that cage.

16. No, it cannot. Let us call a series of faces $F_1, F_2, \dots, F_k$ a *ring* if the pairs (F_1, F_2), (F_2, F_3), $\dots$, (F_{k-1}, F_k), (F_k, F_1) each have a common edge and all these common edges are parallel. It is not difficult to see that any two rings have exactly two common faces and, conversely, each face belongs to exactly two rings. Therefore, if there are n rings then the total number of faces must be $2C_n^2 = n(n-1)$. But there is no positive integer n such that $n(n-1) = 1992$.

17. Denote $\angle ACD = 2\alpha$ (see Fig. 1). Then $\angle CAD = \dfrac{\pi}{2} - 2\alpha$, $\angle ABD = 2\alpha$, $\angle ADB = \dfrac{\pi}{2} - \alpha$ and $\angle CDB = \alpha$. The sine theorem applied to triangles DCP and DAP yields $\dfrac{|DP|}{\sin 2\alpha} = \dfrac{2}{5\sin\alpha}$ and $\dfrac{|DP|}{\sin\left(\frac{\pi}{2} - 2\alpha\right)} = \dfrac{8}{5\sin\left(\frac{\pi}{2} - \alpha\right)}$. Combining these equalities we have $\dfrac{2\sin 2\alpha}{5\sin\alpha} = \dfrac{8\cos 2\alpha}{5\cos\alpha}$ which gives $4\sin\alpha\cos^2\alpha = 8\cos 2\alpha\sin\alpha$ and $\cos 2\alpha + 1 = 4\cos 2\alpha$. So we get $\cos 2\alpha = \dfrac{1}{3}$ and $|CD| = 2\cos 2\alpha = \dfrac{2}{3}$.

18. Let K, L, M be the midpoints of the sides AB, BC, AC of a non-obtuse triangle ABC (see Fig. 2). Note that the centre O of the circumcircle is inside the triangle KLM (or at one of its vertices if ABC is a right-angled triangle). Therefore $|AK|+|KL|+|LC| > |AO|+|OC|$ and hence $|AB|+|AC|+|BC| > 2\cdot(|AO|+|OC|) = 2d$, where d is the diameter of the circumcircle.

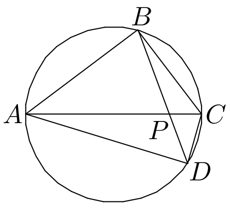

Figure 8

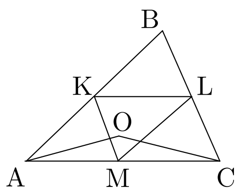

Figure 9

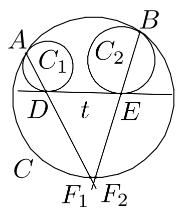

Figure 10

19. Let F_1 be the second intersection point of the line AD and the circle C (see Fig. 3). Consider the homothety with centre A which maps D onto F_1. This homothety maps the circle C_1 onto C and the tangent line t of C_1 onto the tangent line of the circle C at F_1. Let us do the same with the circle C_2 and the line BE: let F_2 be their intersection point and consider the homothety with centre B, mapping E onto F_2, C_2 onto C and t onto the tangent of C at point F_2. Since the tangents of C at F_1 and F_2 are both parallel to t, they must coincide as well as the points F_1 and F_2.

20. By straightforward computation, we find:

$$p(p-c) = \frac{1}{4}\left((a+b)^2 - c^2\right) = \frac{ab}{2} = S\,,$$

$$(p-a)(p-b) = \frac{1}{4}\left(c^2 - (a-b)^2\right) = \frac{ab}{2} = S\,.$$