“Baltic Way – 91” mathematical team contest

Tartu, December 14, 1991

Hints and solutions

1. Let $S = \prod_{1 \leqslant i < j \leqslant n} (a_i - a_j)$. Note that $1991 = 11 \cdot 181$. Therefore S is divisible by 1991 if and only if it is divisible by both 11 and 181 . If $n \leqslant 181$ then we can take the numbers $a_1, \ldots, a_n$ from distinct congruence classes modulo 181 so that S will not be divisible by 181 . On the other hand, if $n \geqslant 182$ then according to the pigeonhole principle there always exist a_i and a_j such that $a_i - a_j$ is divisible by 181 (and of course there exist a_k and a_l such that $a_k - a_l$ is divisible by 11).

2. Factorizing, we get

$$102^{1991} + 103^{1991} = (102 + 103)(102^{1990} - 102^{1989} \cdot 103 + \\ + 102^{1988} \cdot 103^2 + \ldots + 103^{1990})$$

where $102 + 103 = 205 = 5 \cdot 41$. It suffices to show that the other factor is not divisible by 5 . Let $a_k = 102^k \cdot 103^{1990-k}$, then $a_k \equiv 4 \pmod 5$ if k is even and $a_k \equiv -4 \pmod 5$ if k is odd. Thus the whole second factor is congruent to $4 \cdot 1991 \equiv 4 \pmod 5$.

3. The number of different possibilities for buying a cat and a sack is $20 \cdot 20 = 400$ while the number of different possible prices is $1600 - 1210 + 1 = 391$. Thus by the pigeonhole principle there exist two combinations of a cat and a sack costing the same amount of money. Note that the two cats (and also the two sacks) involved must be different as otherwise the two sacks (respectively, cats) would have equal prices.

4. As $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$, then for any distinct integers a, b and for any polynomial $p(x)$ with integer coefficients $p(a) - p(b)$ is divisible by $a - b$. Thus, $p(n) - p(-n) \neq 0$ is divisible by $2n$ and consequently $p(-n) \leqslant p(n) - 2n < n - 2n = -n$.

5. To prove the first inequality, note that $\dfrac{2}{a+b} \leqslant \dfrac{1}{2}\Big(\dfrac{1}{a} + \dfrac{1}{b}\Big)$ and similarly $\dfrac{2}{b+c} \leqslant \dfrac{1}{2}\Big(\dfrac{1}{b} + \dfrac{1}{c}\Big)$, $\dfrac{2}{c+a} \leqslant \dfrac{1}{2}\Big(\dfrac{1}{c} + \dfrac{1}{a}\Big)$. For the second part, use the inequality $\dfrac{3}{x+y+z} \leqslant \dfrac{1}{3}\Big(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\Big)$ for $x = a + b$, $y = b + c$ and $z = c + a$.

6. Denote $f(x) = [x] \cdot \{x\}$, then we have to solve the equation $f(x) = 1991x$. Obviously, $x = 0$ is a solution. For any $x > 0$ we have $0 \leqslant [x] \leqslant x$ and $0 \leqslant \{x\} < 1$ which imply $f(x) < x < 1991x$. For $x \leqslant -1$ we have $0 > [x] > x - 1$ and $0 \leqslant \{x\} < 1$ which imply $f(x) > x - 1 > 1991x$. Finally, if $-1 < x < 0$, then $[x] = -1$, $\{x\} = x - [x] = x + 1$ and $f(x) = -x - 1$. The only solution of the equation $-x - 1 = 1991x$ is $x = -\dfrac{1}{1992}$.

7. In an acute-angled triangle we have $A + B > \dfrac{\pi}{2}$, hence $\sin A > \sin\Big(\dfrac{\pi}{2} - B\Big) = \cos B$ and $\sin B > \cos A$. Using these inequalities we get $(1 - \sin A)(1 - \sin B) < (1 - \cos A)(1 - \cos B)$ and

$$\sin A + \sin B > \cos A + \cos B - \cos A \cos B + \sin A \sin B = \\ = \cos A + \cos B - \cos(A + B) = \\ = \cos A + \cos B + \cos C \ .$$

8. At the left-hand side of the equation we have the derivative of the function $f(x) = (x-a)(x-b)(x-c)(x-d)(x-e)$ which is continuous and has five distinct real roots.

9. Studying the graphs of the functions ae^x and x^3 it is easy to see that the equation has always one solution if $a \leqslant 0$ and can have 0, 1 or 2 solutions if $a > 0$. Moreover, in the case $a > 0$ the number of solutions can only decrease as a increases and we have exactly one positive value of a for which the equation has one solution — this is the case when the graphs of ae^x and x^3 are tangent to each other, i.e. there exists x_0 such that $ae^{x_0} = x_0^3$ and $ae^{x_0} = 3x_0^2$. From these two equations we get $x_0 = 3$ and $a = \dfrac{27}{e^3}$. Summarizing: the equation $ae^x = x^3$ has one solution for $a \leqslant 0$ and $a = \dfrac{27}{e^3}$, two solutions for $0 < a < \dfrac{27}{e^3}$ and no solutions for $a > \dfrac{27}{e^3}$.

10. We use the equality

$$\sin 3^\circ = \sin(18^\circ - 15^\circ) = \sin 18^\circ \cos 15^\circ + \cos 18^\circ \sin 15^\circ$$

where $\sin 15^\circ = \sin\left(\dfrac{30^\circ}{2}\right) = \sqrt{\dfrac{1-\cos 30^\circ}{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos 15^\circ = \sqrt{1-\sin^2 15^\circ} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$. To calculate $\cos 18^\circ$ and $\sin 18^\circ$ note that $\cos(3\cdot 18^\circ) = \sin(2\cdot 18^\circ)$. As $\cos 3x = \cos^3 x - 3\cos x \sin^2 x = \cos x(1-4\sin^2 x)$ and $\sin 2x = 2\sin x \cos x$ we get $1 - 4\sin^2 18^\circ = 2\sin 18^\circ$. Solving this quadratic equation yields $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$ (we discard $\dfrac{-\sqrt{5}-1}{4}$ which is negative) and $\cos 18^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$.

11. Among any ten integers $\overline{a_1 \dots a_n 0}$, $\overline{a_1 \dots a_n 0}$, ..., $\overline{a_1 \dots a_n 0}$ there are exactly five numbers with odd sum of digits and five numbers with even sum of digits. Thus, among the integers $0, 1, \dots, 999\,999$ we have as many numbers with odd sum of digits as there are numbers with even sum of digits. After substituting $1\,000\,000$ instead of 0 we shall have more numbers with odd sum of digits.

12. Assume there exists a renumeration such that for any numbers $1 \leqslant k < l \leqslant n$ the segment connecting vertices numbered k and l before the renumeration has a different colour than the segment connecting vertices with the same numbers after the renumeration. Thus there has to be an equal number of red and blue segments, i.e. the total number of segments should be even. However, the number $C_{1991}^2 = 995 \cdot 1991$ is odd.

13. Define the *distance* between two small triangles to be the minimal number of steps one needs to move from one of the triangles to the other (a step here means transition from one triangle to another having a common side with it). The maximum distance between two small triangles is 8 and this maximum is achieved if and only if one of these lies at a corner of the big triangle and the other lies anywhere at the opposite side of it. Assume now that we have assigned the numbers $1, \dots, 25$ to the small triangles so that the difference of the numbers assigned to any two adjacent triangles does not exceed 3. Then the distance between the triangles numbered 1 and 25; 1 and 24; 2 and 25; 2 and 24 must be equal to 8. However, this is not possible since it implies that either the numbers 1 and 2 or 24 and 25 should be assigned to the same "corner" triangle.

14. The knight can use the following strategy: exit from any hall through the door immediately to the right of the one he used to enter that hall. Then, knowing which door was passed last and in which direction we can uniquely restore the whole path of the knight up to that point. Therefore, he will not be able to pass any door twice in the same direction unless he has been outside the castle in between.

15. Figure 1 demonstrates a possible king's path passing through each square exactly once and finally returning to the initial square. Thus, it suffices to prove part c) as we can always increase the numbers on all the squares by 1 or 2 if necessary. Moreover, note that for any given square it is possible to modify the path shown on Fig. 1 in such a way that this particular square will be passed twice while any other square will still be passed exactly once. Repeating this procedure a suitable number of times for each square we can make all the numbers on the chessboard equal to each other.

16. Let P_1, P_2, P_3 be the perpendicular projections of O_1, O_2, O_3 to the line l and let Q be the perpendicular projection of O_3 to the line P_1O_1 (see Fig. 2). Then $|QO_3|^2 = |O_1O_3|^2 - |QO_1|^2$ and $|P_1P_3|^2 = (r_1 + r_3)^2 - (r_1 - r_3)^2 = 4r_1r_3$. Similarly we get $|P_1P_2|^2 = 4r_1r_2$ and $|P_2P_3|^2 = 4r_2r_3$. Since $|P_1P_2| = |P_1P_3| + |P_2P_3|$ we have $\sqrt{r_1r_2} = \sqrt{r_1r_3} + \sqrt{r_2r_3}$ which clearly implies the required equality.

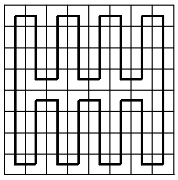

Figure 5

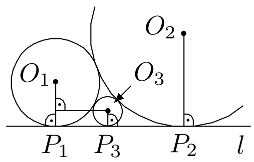

Figure 6

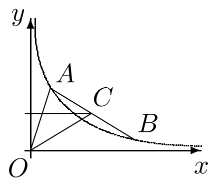

Figure 7

17. Let the velocity vector of the plane be $\vec{v} = (\alpha, \beta, \gamma)$. Reflection from each of the coordinate planes changes the sign of exactly one of the coordinates α, β and γ, thus the final direction will be opposite to the initial one.

18. No, it is not. Any tetrahedron that does not contain the centre of the sphere as an internal point has a height drawn to one of its faces less or equal than the radius of the sphere, i.e. 1. As each of the faces of the tetrahedron is contained in a circle with radius not greater than 1, its area cannot exceed $\frac{3\sqrt{3}}{4}$. Thus, the volume of such a tetrahedron must be less or equal than $\frac{1}{3} \cdot 1 \cdot \frac{3\sqrt{3}}{4} = \frac{\sqrt{3}}{4} < \frac{1}{2}$.

19. First, note that the three straight lines A_1A_2, B_1B_2 and C_1C_2 intersect in a single point O. Indeed, each of the lines is the locus of points from which the tangents to two of the circles are of equal length (it is easy to check that this locus has the form of a straight line and obviously it contains the two intersection points of the circles). Now, we have $|OA_1| \cdot |OA_2| = |OB_1| \cdot |OB_2|$ (as both of these products are equal to $|OT|^2$ where OT is a tangent line to the circle containing A_1, A_2, B_1, B_2 and T is the corresponding point of tangency). Hence $\frac{|OA_1|}{|OB_2|} = \frac{|OB_1|}{|OA_2|}$ which implies that the triangles OA_1B_2 and OB_1A_2 are similar and $\frac{|A_1B_2|}{|A_2B_1|} = \frac{|OA_1|}{|OB_1|}$. Similarly we get $\frac{|B_1C_2|}{|B_2C_1|} = \frac{|OB_1|}{|OC_1|}$ and $\frac{|C_1A_2|}{|C_2A_1|} = \frac{|OC_1|}{|OA_1|}$. It remains to multiply these three equalities.

20. We have $A\Big(x_1, \frac{1}{x_1}\Big)$, $B\Big(x_2, \frac{1}{x_2}\Big)$ and $C\Big(\frac{x_1 + x_2}{2}, \frac{1}{2x_1} + \frac{1}{2x_2}\Big)$. Computing the coordinates of $\vec{v} = |OC| \cdot \overrightarrow{AC} + |AC| \cdot \overrightarrow{OC}$ we find that the vector $\vec{v}$ — and hence also the bisector of the angle $\angle OCA$ — is parallel to the x-axis. Since $|OA| = |AC|$ this yields $\angle AOC = \angle ACO = 2 \cdot \angle COx$ (see Fig. 3) and $\angle AOx = \angle AOC + \angle COx = 3 \cdot \angle COx$.