“Baltic Way – 90” mathematical team contest

Riga, November 24, 1990

Hints and solutions

1. Let $a_1 = 1,\, a_2,\, \ldots,\, a_k = n,\, a_{k+1},\, \ldots,\, a_n$ be the order in which the numbers $1, 2, \ldots, n$ are written around the circle, then the sum of moduli of the differences of the neighbouring numbers is

$$\begin{aligned} &|1-a_2|+|a_2-a_3|+\ldots+|a_k-n|+|n-a_{k+1}|+\ldots+|a_n-1| \geqslant \\ &\quad \geqslant |1-a_2+a_2-a_3+\ldots+a_k-n|+|n-a_{k+1}+\ldots+a_n-1| = \\ &\quad = |1-n|+|n-1| = 2n-2\,. \end{aligned}$$

This minimum is achieved if the numbers are written around the circle in increasing order.

2. Since the square with the coordinates (m,n) is n-th on the $(n+m-1)$-th diagonal, it contains the number

$$P(m,n) = \sum_{i=1}^{n+m-2} i + n = \frac{(n+m-1)(n+m-2)}{2} + n\,.$$

3. Obviously we can find angles $0 < \alpha, \beta < 90^\circ$ such that $\tan\alpha > 0$, $\tan(\alpha+\beta) > 0$, $\ldots$, $\tan(\alpha+1989\beta) > 0$ but $\tan(\alpha+1990\beta) < 0$. Now it suffices to note that if we take $a_0 = \tan\alpha$ and $c = \tan\beta$ then $a_n = \tan(\alpha+n\beta)$.

4. Consider the polynomial $P(x) = a_1 + a_2 x + \ldots + a_n x^{n-1}$, then $P^2(x) = \sum_{k,l=1}^{n} a_k a_l x^{k+l-2}$ and

$$\int_0^1 P^2(x) = \sum_{k,l=1}^{n} \frac{a_k a_l}{k+l-1}\quad.$$

5. A suitable equation is $x*(x*x) = (x*x)*x$ which is obviously true if $*$ is any commutative or associative operation but does not hold in general, e.g. $1-(1-1) \neq (1-1)-1$.

6. Note that $\angle ADC + \angle CDP + \angle BCD + \angle DCP = 360^\circ$ (see Fig. 1). Thus $\angle ADP = 360^\circ - \angle BCD - \angle DCP = \angle BCP$. As we have $|DP| = |CP|$ and $|AD| = |BC|$ then the triangles ADP and BCP are congruent and $|AP| = |BP|$. Moreover, $\angle APB = 60^\circ$ since $\angle DPC = 60^\circ$ and $\angle DPA = \angle CPB$.

7. It suffices to show that the centre of gravity of the pentagon (viewed as a system of five equal masses placed at its vertices) lies on each of the five segments. To prove that, divide this system of masses into two subsystems, one of which consists of the two masses at the endpoints of the side under consideration and the other consists of the three remaining masses at the vertices of the triangle. The segment mentioned in the problem connects the centres of gravity of these two subsystems, hence it contains the centre of gravity of the whole system.

8. Let O be the circumcentre of the triangle ABC and $\angle B$ be its maximal angle (so that $\angle A$ and $\angle C$ are necessarily acute). Further, let B_1 and C_1 be the basepoints of the perpendiculars drawn from the point P to the sides AC and AB respectively and let α be the angle between the Simpson line l of point P and the height h of the triangle drawn to the side AC. It is sufficient to prove that $\alpha = \frac{1}{2}\angle POB$. To show this, first note that the points P, C_1, B_1, A all belong to a certain circle. Now we have to consider several subcases depending on the order of these points on that circle and the location of point P on the circumcircle of triangle ABC. Fig. 2 shows one of these cases — here we have $\alpha = \angle PB_1C_1 = \angle PB_1C_1 = \angle PAB = \frac{1}{2}\angle POB$. The other cases can be treated in a similar manner.

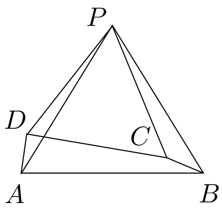

Figure 1

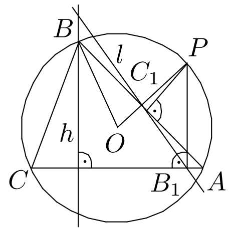

Figure 2

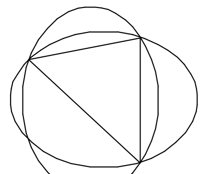

Figure 3

9. No, not necessarily (see Fig. 3 where the two ellipses are equal).

10. The point A can move to any distance from its initial position — see Fig. 4 and note that we can make the height h arbitrarily small.

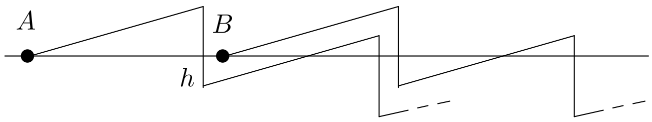

Figure 4

11. For a polynomial $P(x) = a_n x^n + \ldots + a_1 x + a_0$ with integer coefficients let k be the smallest index such that $a_k \neq 0$, i.e. in fact $P(x) = a_n x^n + \ldots + a_k x^k$. If c is an integer root of $P(x)$ then $P(x) = (x - c) \cdot Q(x)$ where $Q(x) = b_{n-1}x^{n-1} + \ldots + b_k x^k$ is a polynomial with integer coefficients, $a_n = b_{n-1}$ and $a_k = -b_k \cdot c$. Since $a_k \neq 0$ we have $b_k \neq 0$ and $|c| \leqslant |a_k|$.

12. Use the equality $2 \cdot (25x + 3y) + 11 \cdot (3x + 7y) = 83x + 83y$.

13. For any solution (m, n) of the equation we have $m^2 - 7n^2 = 1$ and $1 = (m^2 - 7n^2)^2 = (m^2 + 7n^2)^2 - 7 \cdot (2mn)^2$, hence $(m^2 + 7n^2, 2mn)$ is also a solution. Therefore it is sufficient to note that the equation $x^2 - 7y^2 = 1$ has at least one solution, e.g. $x = 8$, $y = 3$.

14. Such numbers do exist. Let $M = 1990!$ and consider the sequence of numbers $1+M$, $1+2M$, $1+3M$, $\ldots$. For any natural number $2 \leqslant k \leqslant 1990$ any sum S of exactly k of these numbers (not necessarily different) is divisible by $k < S$ and hence a composite number. It remains to show that we can choose 1990 numbers $a_1, \ldots, a_{1990}$ from this sequence which are relatively prime. Indeed, let $a_1 = 1 + M$, $a_2 = 1 + 2M$ and for $a_1, \ldots, a_n$ already chosen take $a_{n+1} = 1 + a_1 \cdot \ldots \cdot a_n \cdot M$.

15. Assume there exist such natural numbers k and n that $2^{2^n} + 1 = k^3$. Then k must be an odd number and we have $2^{2^n} = k^3 - 1 = (k-1)(k^2 + k + 1)$. Hence $k - 1 = 2^s$ and $k^2 + k + 1 = 2^t$ where s and t are some natural numbers. Now $2^{2s} = (k-1)^2 = k^2 - 2k + 1$ and $2^t - 2^{2s} = 3k$, but $2^t - 2^{2s}$ is even while $3k$ is odd, a contradiction.

16. There must be an equal number of horizontal and vertical links, hence it suffices to show that the number of vertical links is even. Let's pass the whole polygonal line in a chosen direction and mark each vertical link as "up" or "down" according to the direction we pass it. As the sum of lengths of the "up" links is equal to that of the "down" ones and each link is of odd length then we have an even or odd number of links of both kinds depending on the parity of the sum of their lengths.

17. Note that one of the players must have a "winning" strategy. Assume it is the player making the second move who has it, then his strategy will assure taking the last sweet also in the case when the beginner takes $2 \cdot 30$ sweets as his first move. But now, if the beginner takes $1 \cdot 30$ sweets then the second player has no choice but to take another 30 sweets from the same pile, and hence the beginner can use the same strategy to assure taking the last sweet himself. This contradiction shows that it must be the beginner who has the "winning" strategy.

18. Let a_k denote the total number of rows and columns containing the number k at least once. As $i \cdot (20 - i) < 101$ for any natural number i we have $a_k \geqslant 21$ for all $k = 1, 2, \ldots, 101$. Hence $a_1 + \ldots + a_{101} \geqslant 21 \cdot 101 = 2121$. On the other hand, assuming any row and column contains no more than 10 different numbers we have $a_1 + \ldots + a_{101} \leqslant 202 \cdot 10 = 2020$, a contradiction.

19. Consider any subsets $A_1, \ldots, A_s$ satisfying the condition of the problem and let $A_i = \{ a_{i1}, \ldots, a_{i,k_i} \}$ where $a_{i1} < \ldots < a_{i,k_i}$. Replacing each A_i by $A'_i = \{ a_{i1}, a_{i1} + 1, \ldots, a_{i,k_i} - 1, a_{i,k_i} \}$ (i.e. adding to it all "missing" numbers) yields a collection of different subsets $A'_1, \ldots, A'_s$ which also satisfies the required condition. Now, let b_i and c_i be the smallest and largest elements of the subset A'_i respectively, then $\min\limits_{1 \leqslant i \leqslant s} c_i \geqslant \max\limits_{1 \leqslant i \leqslant s} b_i$ as otherwise some subsets A'_k and A'_l would not intersect. Hence there exists an element $a \in \bigcap\limits_{1 \leqslant i \leqslant s} A'_i$. As the number of subsets of the set $\{ 1, 2, \ldots, 2n + 1 \}$ containing a and consisting of k consequtive integers does not exceed $\min(k, 2n + 2 - k)$ we have $s \leqslant (n + 1) + 2 \cdot (1 + 2 + \ldots + n) = (n + 1)^2$. This maximum will be reached if we take $a = n + 1$.