Baltic Way

16 November 2024, Tartu, Estonia

Version: English

Time allowed: 4 hours and 30 minutes
Questions may be asked during the first 30 minutes.
Tools for writing and drawing are the only ones allowed.

Problem 1 Let α be a non-zero real number. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x+y) = (x+\alpha y)f(x) + xf(y)$$

for all $x, y \in \mathbb{R}$.

Problem 2 Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{f(a)}{1+a+ca} + \frac{f(b)}{1+b+ab} + \frac{f(c)}{1+c+bc} = 1$$

for all $a, b, c \in \mathbb{R}^+$ that satisfy $abc = 1$.

Problem 3 Positive real numbers $a_1, a_2, \ldots, a_{2024}$ are written on the blackboard. A move consists of choosing two numbers x and y on the blackboard, erasing them and writing the number $\dfrac{x^2+6xy+y^2}{x+y}$ on the blackboard. After 2023 moves, only one number c will remain on the blackboard. Prove that

$$c < 2024\left(a_1 + a_2 + \ldots + a_{2024}\right).$$

Problem 4 Find the largest real number α such that, for all non-negative real numbers x, y and z, the following inequality holds:

$$(x+y+z)^3 + \alpha\left(x^2z + y^2x + z^2y\right) \geq \alpha\left(x^2y + y^2z + z^2x\right).$$

Problem 5 Find all positive real numbers λ such that every sequence $a_1, a_2, \ldots$ of positive real numbers satisfying

$$a_{n+1} = \lambda \cdot \frac{a_1 + a_2 + \ldots + a_n}{n}$$

for all $n \geq 2024^{2024}$ is bounded.

Remark: A sequence $a_1, a_2, \ldots$ of positive real numbers is *bounded* if there exists a real number M such that $a_i < M$ for all $i = 1, 2, \ldots$

Problem 6 A *labyrinth* is a system of 2024 caves and 2023 non-intersecting (bidirectional) corridors, each of which connects exactly two caves, where each pair of caves is connected through some sequence of corridors. Initially, Erik is standing in a corridor connecting some two caves. In a move, he can walk through one of the caves to another corridor that connects that cave to a third cave. However, when doing so, the corridor he was just in will magically disappear and get replaced by a new one connecting the end of his new corridor to the beginning of his old one (i.e., if Erik was in a corridor connecting caves a and b and he walked through cave b into a corridor that connects caves b and c, then the corridor between caves a and b will disappear and a new corridor between caves a and c will appear).

Since Erik likes designing labyrinths and has a specific layout in mind for his next one, he is wondering whether he can transform the labyrinth into that layout using these moves. Prove that this is in fact possible, regardless of the original layout and his starting position there.

Problem 7 A 45×45 grid has had the central unit square removed. For which positive integers n is it possible to cut the remaining area into $1 \times n$ and $n \times 1$ rectangles?

Problem 8 Let a, b, n be positive integers such that $a + b \leq n^2$. Alice and Bob play a game on an (initially uncoloured) $n \times n$ grid as follows:

- First, Alice paints a cells green.
- Then, Bob paints b other (i.e. uncoloured) cells blue.

Alice wins if she can find a path of non-blue cells starting with the bottom left cell and ending with the top right cell (where a path is a sequence of cells such that any two consecutive ones have a common side), otherwise Bob wins. Determine, in terms of a, b and n, who has a winning strategy.

Problem 9 Let S be a finite set. For a positive integer n, we say that a function $f\colon S \to S$ is an *n-th power* if there exists some function $g\colon S \to S$ such that

$$f(x) = \underbrace{g(g(\ldots g(x)\ldots))}_{g \text{ applied } n \text{ times}}$$

for each $x \in S$.

Suppose that a function $f\colon S \to S$ is an n-th power for each positive integer n. Is it necessarily true that $f(f(x)) = f(x)$ for each $x \in S$?

Problem 10 A frog is located on a unit square of an infinite grid oriented according to the cardinal directions. The frog makes moves consisting of jumping either one or two squares in the direction it is facing, and then turning according to the following rules:

(i) If the frog jumps one square, it then turns $90°$ to the right;

(ii) If the frog jumps two squares, it then turns $90°$ to the left.

Is it possible for the frog to reach the square exactly 2024 squares north of the initial square after some finite number of moves if it is initially facing:

(a) North;

(b) East?

Problem 11 Let $ABCD$ be a cyclic quadrilateral with circumcentre O and with AC perpendicular to BD. Points X and Y lie on the circumcircle of the triangle BOD such that $\angle AXO = \angle CYO = 90°$. Let M be the midpoint of AC. Prove that BD is tangent to the circumcircle of the triangle MXY.

Problem 12 Let ABC be an acute triangle with circumcircle ω such that $AB < AC$. Let M be the midpoint of the arc BC of ω containing the point A, and let $X \neq M$ be the other point on ω such that $AX = AM$. Points E and F are chosen on sides AC and AB of the triangle ABC, respectively, such that $EX = EC$ and $FX = FB$. Prove that $AE = AF$.

Problem 13 Let ABC be an acute triangle with orthocentre H. Let D be a point outside the circumcircle of triangle ABC such that $\angle ABD = \angle DCA$. The reflection of AB in BD intersects CD at X. The reflection of AC in CD intersects BD at Y. The lines through X and Y perpendicular to AC and AB, respectively, intersect at P. Prove that points D, P and H are collinear.

Problem 14 Let ABC be an acute triangle with circumcircle ω. The altitudes AD, BE and CF of the triangle ABC intersect at point H. A point K is chosen on the line EF such that $KH \parallel BC$. Prove that the reflection of H in KD lies on ω.

Problem 15 There is a set of $N \geq 3$ points in the plane, such that no three of them are collinear. Three points A, B, C in the set are said to form a *Baltic triangle* if no other point in the set lies on the circumcircle of triangle ABC. Assume that there exists at least one Baltic triangle.

Show that there exist at least $\dfrac{N}{3}$ Baltic triangles.

Problem 16 Determine all composite positive integers n such that, for each positive divisor d of n, there are integers $k \geq 0$ and $m \geq 2$ such that $d = k^m + 1$.

Problem 17 Do there exist infinitely many quadruples (a, b, c, d) of positive integers such that the number $a^{a!} + b^{b!} - c^{c!} - d^{d!}$ is prime and $2 \leq d \leq c \leq b \leq a \leq d^{2024}$?

Problem 18 An infinite sequence $a_1, a_2, \ldots$ of positive integers is such that $a_n \geq 2$ and a_{n+2} divides $a_{n+1} + a_n$ for all $n \geq 1$. Prove that there exists a prime which divides infinitely many terms of the sequence.

Problem 19 Does there exist a positive integer N which is divisible by at least 2024 distinct primes and whose positive divisors $1 = d_1 < d_2 < \ldots < d_k = N$ are such that the number

$$\frac{d_2}{d_1} + \frac{d_3}{d_2} + \ldots + \frac{d_k}{d_{k-1}}$$

is an integer?

Problem 20 Positive integers a, b and c satisfy the system of equations

$$\begin{cases} (ab-1)^2 = c\left(a^2+b^2\right) + ab + 1, \\ a^2 + b^2 = c^2 + ab. \end{cases}$$

(a) Prove that $c + 1$ is a perfect square.

(b) Find all such triples (a, b, c).