BALTIC WAY 2016 OULU

Baltic Way 2016
5 November 2016, Oulu, Finland
Version: English

Working time: $4\frac{1}{2}$ hours. Questions may be asked during the first 30 minutes.
Tools for writing and drawing are the only ones allowed.

1. Find all pairs of primes (p, q) such that

$$p^3 - q^5 = (p+q)^2.$$

2. Prove or disprove the following hypotheses.

a) For all $k \geq 2$, each sequence of k consecutive positive integers contains a number that is not divisible by any prime number less than k.

b) For all $k \geq 2$, each sequence of k consecutive positive integers contains a number that is relatively prime to all other members of the sequence.

3. For which integers $n = 1, \ldots, 6$ does the equation

$$a^n + b^n = c^n + n$$

have a solution in integers?

4. Let n be a positive integer and let a, b, c, d be integers such that $n \mid a+b+c+d$ and $n \mid a^2+b^2+c^2+d^2$. Show that

$$n \mid a^4+b^4+c^4+d^4+4abcd.$$

5. Let $p > 3$ be a prime such that $p \equiv 3 \pmod 4$. Given a positive integer a_0 define the sequence $a_0, a_1, \ldots$ of integers by $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \ldots$. Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is not constant modulo p for any positive integer N.

6. The set $\{1, 2, \ldots, 10\}$ is partitioned to three subsets A, B and C. For each subset the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated.

Is it possible that A alone has the largest sum of elements, B alone has the largest product of elements, and C alone has the largest sum of digits?

7. Find all positive integers n for which

$$3x^n + n(x+2) - 3 \geq nx^2$$

holds for all real numbers x.

8. Find all real numbers a for which there exists a non-constant function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the following two equations for all $x \in \mathbb{R}$:

i) $f(ax) = a^2 f(x)$ and

ii) $f(f(x)) = a\, f(x)$.

9. Find all quadruples (a, b, c, d) of real numbers that simultaneously satisfy the following equations:

$$\begin{cases} a^3 &+& c^3 &=& 2 \\ a^2b &+& c^2d &=& 0 \\ b^3 &+& d^3 &=& 1 \\ ab^2 &+& cd^2 &=& -6. \end{cases}$$

10. Let $a_{0,1}$, $a_{0,2}$, $\ldots$, $a_{0,2016}$ be positive real numbers. For $n \geq 0$ and $1 \leq k < 2016$ set

$$a_{n+1,k} = a_{n,k} + \frac{1}{2a_{n,k+1}} \quad \text{and} \quad a_{n+1,2016} = a_{n,2016} + \frac{1}{2a_{n,1}}.$$

Show that $\max\limits_{1 \leq k \leq 2016} a_{2016,k} > 44$.

11. Set A consists of 2016 positive integers. All prime divisors of these numbers are smaller than 30. Prove that there are four distinct numbers a, b, c and d in A such that $abcd$ is a perfect square.

12. Does there exist a hexagon (not necessarily convex) with side lengths 1, 2, 3, 4, 5, 6 (not necessarily in this order) that can be tiled with a) 31 b) 32 equilateral triangles with side length 1?

13. Let n numbers all equal to 1 be written on a blackboard. A move consists of replacing two numbers on the board with two copies of their sum. It happens that after h moves all n numbers on the blackboard are equal to m. Prove that $h \leq \frac{1}{2} n \log_2 m$.

14. A cube consists of 4^3 unit cubes each containing an integer. At each move, you choose a unit cube and increase by 1 all the integers in the neighbouring cubes having a face in common with the chosen cube. Is it possible to reach a position where all the 4^3 integers are divisible by 3, no matter what the starting position is?

15. The Baltic Sea has 2016 harbours. There are two-way ferry connections between some of them. It is impossible to make a sequence of direct voyages $C_1 - C_2 - \cdots - C_{1062}$ where all the harbours $C_1, \ldots, C_{1062}$ are distinct. Prove that there exist two disjoint sets A and B of 477 harbours each, such that there is no harbour in A with a direct ferry connection to a harbour in B.

16. In triangle ABC, the points D and E are the intersections of the angular bisectors from C and B with the sides AB and AC, respectively. Points F and G on the extensions of AB and AC beyond B and C, respectively, satisfy $BF = CG = BC$. Prove that $FG \parallel DE$.

17. Let $ABCD$ be a convex quadrilateral with $AB = AD$. Let T be a point on the diagonal AC such that $\angle ABT + \angle ADT = \angle BCD$. Prove that $AT + AC \geq AB + AD$.

18. Let $ABCD$ be a parallelogram such that $\angle BAD = 60^\circ$. Let K and L be the midpoints of BC and CD, respectively. Assuming that $ABKL$ is a cyclic quadrilateral, find $\angle ABD$.

19. Consider triangles in the plane where each vertex has integer coordinates. Such a triangle can be *legally transformed* by moving one vertex parallel to the opposite side to a different point with integer coordinates. Show that if two triangles have the same area, then there exists a series of legal transformations that transforms one to the other.

20. Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.