16^{th} Baltic Way Mathematical Team Contest: Solutions

November 5, 2005, Stockholm, Sweden

1. **Answer:** No, the sequence must contain two equal terms.

   It is clear that there exists a smallest positive integer k such that

   $$10^k > (k+1) \cdot 9^{2005}.$$

   We shall show that there exists a positive integer N such that a_n consists of less than $k+1$ decimal digits, $\forall n \geq N$. Let a_i be a positive integer which consists of exactly $j+1$ digits, that is,

   $$10^j \leq a_i < 10^{j+1}.$$

   We need to prove two statements:

   - a_{i+1} has less than $k+1$ digits if $j < k$, and
   - $a_i > a_{i+1}$ if $j \geq k$.

   To prove the first statement, notice that

   $$a_{i+1} \leq (j+1) \cdot 9^{2005} < (k+1) \cdot 9^{2005} < 10^k \quad (j < k)$$

   and hence a_{i+1} consists of less than $k+1$ digits. To prove the second statement, notice that a_i consists of $j+1$ digits, none of which exceeds 9. Hence $a_{i+1} \leq (j+1) \cdot 9^{2005}$ and because $j \geq k$, we get

   $$a_i \geq 10^j > (j+1) \cdot 9^{2005} \geq a_{i+1} \quad (j \geq k),$$

   which proves the second statement. It is now easy to derive the result from this statement. Assume that a_0 consists of $k+1$ or more digits (otherwise we are done, because then it follows inductively that all terms of the sequence consist of less than $k+1$ digits, by the first statement). It is obviously possible to construct a strictly decreasing sequence $a_0 > a_1 > \cdots > a_N$ of positive integers such that a_N has less than $k+1$ digits (where N is the first index having this property). By an easy induction, it follows that none of the numbers in $\{a_N,\ a_{N+1}, \ldots\}$ consists of more than k digits. This set contains infinitely many numbers but none of these numbers exceeds 10^k. By the Pigeonhole Principle, two elements of this set must be equal, and we are done.

2. Since $\tan^2 x = 1/\cos^2 x - 1$, the inequality to be proved is equivalent to

   $$\frac{1}{\cos^2\alpha} + \frac{1}{\cos^2\beta} + \frac{1}{\cos^2\gamma} \geq \frac{27}{8}.$$

   The inequality between arithmetic and harmonic means implies

   $$\begin{aligned}\frac{3}{\frac{1}{\cos^2\alpha} + \frac{1}{\cos^2\beta} + \frac{1}{\cos^2\gamma}} &\leq \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}{3} \\ &= \frac{3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{3} \\ &\leq 1 - \left(\frac{\sin\alpha + \sin\beta + \sin\gamma}{3}\right)^2 \\ &= \frac{8}{9}\end{aligned}$$

   and the result follows.

3. Note that
$$\frac{1}{a_k a_{k+2}} < \frac{2}{a_k a_{k+1}} - \frac{2}{a_{k+1} a_{k+2}}\,,$$
because this inequality is equivalent to the inequality
$$a_{k+2} > a_k + \frac{1}{2} a_{k+1}\,,$$
which is evident for the given sequence. Now we have
$$\frac{1}{a_1 a_3} + \frac{1}{a_2 a_4} + \frac{1}{a_3 a_5} + \cdots + \frac{1}{a_{98} a_{100}} < \frac{2}{a_1 a_2} - \frac{2}{a_2 a_3} + \frac{2}{a_2 a_3} - \frac{2}{a_3 a_4} + \cdots < \frac{2}{a_1 a_2} = 4\,.$$

4. **Answer:** For example, $P(x) = x$, $P(x) = x^2 + 1$ and $P(x) = x^4 + 2x^2 + 2$.

   For all reals x we have $P(x)^2 + 1 = P(x^2 + 1) = P(-x)^2 + 1$ and consequently, $(P(x) + P(-x))(P(x) - P(-x)) = 0$. Now one of the three cases holds:

   (a) $P(x)+P(-x) \not\equiv 0$ and $P(x)-P(-x) \not\equiv 0$. Then $(P(x)+P(-x))$ as well as $(P(x)-P(-x))$ are both non-constant polynomials and have a finite numbers of roots only, i.e. this case cannot hold.

   (b) $P(x) + P(-x) \equiv 0$. Obviously, $P(0) = 0$. Consider the infinite sequence of integers $a_0 = 0$ and $a_{n+1} = a_n^2 + 1$. By induction it is easy to see that $P(a_n) = a_n$ for all non-negative integers n. Also, $Q(x) = x$ has that property, i.e. $P(x) - Q(x)$ is a polynomial with infinitely many roots, i.e. $P(x) \equiv x$.

   (c) $P(x) - P(-x) \equiv 0$. Then
   $$P(x) = x^{2n} + b_{n-1}x^{2n-2} + \ldots + b_1 x^2 + b_0,$$
   for some integer n since $P(x)$ is even and it is easy to see that the coefficient of x^{2n} must be 1. $n = 1$ and $n = 2$ yield the solutions $P(x) = x^2 + 1$ and $P(x) = x^4 + 2x^2 + 2$.

   Remark: For $n = 3$ there is no solution, whereas for $n = 4$ there is the unique solution $P(x) = x^8 + 6x^6 + 8x^4 + 8x^2 + 5$.

   Alternative solution: Let $Q(x) = x^2 + 1$. Then the equation that P must satisfy can be written $P(Q(x)) = Q(P(x))$, and it is clear that this will be satisfied for $P(x) = x$, $P(x) = Q(x)$ and $P(x) = Q(Q(x))$.

5. For any positive real x we have $x^2 + 1 \geq 2x$. Hence
$$\frac{a}{a^2+2} + \frac{b}{b^2+2} + \frac{c}{c^2+2} \leq$$
$$\frac{a}{2a+1} + \frac{b}{2b+1} + \frac{c}{2c+1} = \frac{1}{2+1/a} + \frac{1}{2+1/b} + \frac{1}{2+1/c} = R.$$
$R \leq 1$ is equivalent to
$$\left(2+\frac{1}{b}\right)\left(2+\frac{1}{c}\right) + \left(2+\frac{1}{a}\right)\left(2+\frac{1}{c}\right) + \left(2+\frac{1}{a}\right)\left(2+\frac{1}{b}\right) \leq \left(2+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(2+\frac{1}{c}\right)$$
and to $4 \leq \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{1}{abc}$. By $abc = 1$ and by the AM-GM inequality
$$\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \geq 3\sqrt[3]{\left(\frac{1}{abc}\right)^2} = 3$$
the last inequality follows. Equality appears exactly when $a = b = c = 1$.

6. Let $N = q \cdot K + r$, $0 \le r < K$, and let us number the cards $1, 2, \ldots, N$, starting from the one at the bottom of the deck. First we find out how the cards $1, 2, \ldots, K$ are moving in the deck.

   If $i \le r$ then the card i is moving along the cycle

$$i, K + i, 2K + i, \ldots, qK + i, (r + 1 - i), K + (r + 1 - i), \ldots, qK + (r + 1 - i),$$

   because $N - K < qK + i \le N$ and $N - K < qK + (r + 1 - i) \le N$. The length of this cycle is $2q + 2$. In a special case of $i = r + i - 1$, it actually consists of two smaller cycles of length $q + 1$.

   If $r < i \le K$ then the card i is moving along the cycle

$$\begin{aligned} &i, K + i, 2K + i, \ldots, (q - 1)K + i, (K + r + 1 - i), \\ &K + (K + r + 1 - i), 2K + (K + r + 1 - i), \ldots, (q - 1)K + (K + r + 1 - i), \end{aligned}$$

   because $N - K < (q - 1)K + i \le N$ and $N - K < (q - 1)K + (K + r + 1 - i) \le N$. The length of this cycle is $2q$. In a special case of $i = K + r + 1 - i$, it actually consists of two smaller cycles of length q.

   Since these cycles cover all the numbers $1, \ldots, N$, we can say that every card returns to its initial position after either $2q+2$ or $2q$ operations. Therefore, all the cards are simultaneously at their initial position after no later than $\mathrm{LCM}(2q + 2, 2q) = 2\,\mathrm{LCM}(q + 1, q) = 2q(q + 1)$ operations. Finally,

$$2q(q + 1) \le (2q)^2 = 4q^2 \le 4\left(\frac{N}{K}\right)^2,$$

   which concludes the proof.

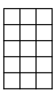
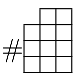
7. Clearly there must be rows with some zeroes. Consider the case when there is a row with just one zero; we can assume it is $(0, 1, 1, 1, 1, 1)$. Then for each row $(1, x_2, x_3, x_4, x_5, x_6)$ there is also a row $(0, x_2, x_3, x_4, x_5, x_6)$; the conclusion follows. Consider the case when there is a row with just two zeros; we can assume it is $(0, 0, 1, 1, 1, 1)$. Let n_{ij} be the number of rows with first two elements i, j. As in the first case $n_{00} \ge n_{11}$. Let $n_{01} \ge n_{10}$; the other subcase is analogous. Now there are $n_{00} + n_{01}$ zeros in the first column and $n_{10} + n_{11}$ ones in the first column; the conclusion follows. Consider now the case when each row contains at least 3 zeros (except $(1, 1, 1, 1, 1, 1)$, if such a row exists). Let's prove it is impossible that each such row contains exactly 3 zeroes. Assume the opposite. As $n > 2$ there are at least 2 rows with zeros; they are different, so their product contains at least 4 zeros, a contradiction. So there are more then $3(n - 1)$ zeros in the array; so in some column there are more than $(n - 1)/2$ zeros; so there are at least $n/2$ zeros.

8. **Answer:** 48 squares.

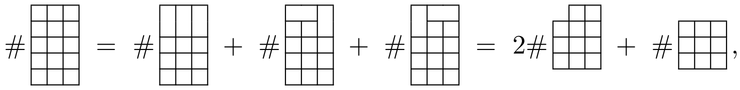
   Consider a diagonal of the square grid. For any grid vertex A on this diagonal denote by C the farthest endpoint of this diagonal. Let the square with the diagonal AC be red. Thus, we have defined the set of 48 red squares (24 for each diagonal). It is clear that if we draw all these squares, all the lines in the grid will turn red.

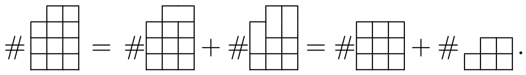
   In order to show that 48 is the minimum, consider all grid segments of length 1 that have exactly one endpoint on the border of the grid. Every horizontal and every vertical line that cuts the grid into two parts determines two such segments. So we have $4 \cdot 24 = 96$ segments. It is evident that every red square can contain at most 2 of these segments.

9. Let us denote the number of ways to split some figure onto dominos by a small picture of this figure with a sign #. For example, # = 2.

   Let N_n =# (n rows); γ_n =# ($n - 2$ full rows and 1 row with 2 cells). We are going to find a recurrent relation for the numbers N_n.

Observe that





We can generalize our observations by writing the equalities

$$\begin{aligned} N_n &= 2\gamma_n + N_{n-2}\,, \\ 2\gamma_{n-2} &= N_{n-2} - N_{n-4}\,, \\ 2\gamma_n &= 2\gamma_{n-2} + 2N_{n-2}\,. \end{aligned}$$

If we sum up these equalities we obtain the desired recurrence

$$N_n = 4N_{n-2} - N_{n-4}\,.$$

It is easy to find that $N_2 = 3$, $N_4 = 11$. Now by the recurrence relation it is trivial to check that $N_{6k+2} \equiv 0 \pmod 3$.

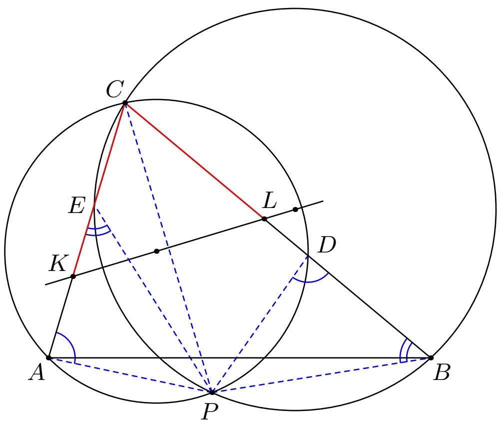
10. **Answer:** $n = 11$.

Taking the 10 divisors without prime 13 shows $n \geq 11$. Consider the following partition of the 15 divisors into 5 groups of 3 each with the property that the product of the numbers in every group equals m.

$$\begin{gathered} \{2\cdot 3,\ 5\cdot 13,\ 7\cdot 11\}, \\ \{2\cdot 5,\ 3\cdot 7,\ 11\cdot 13\}, \\ \{2\cdot 7,\ 3\cdot 13,\ 5\cdot 11\}, \\ \{2\cdot 11,\ 3\cdot 5,\ 7\cdot 13\}, \\ \{2\cdot 13,\ 3\cdot 11,\ 5\cdot 7\} \end{gathered}$$

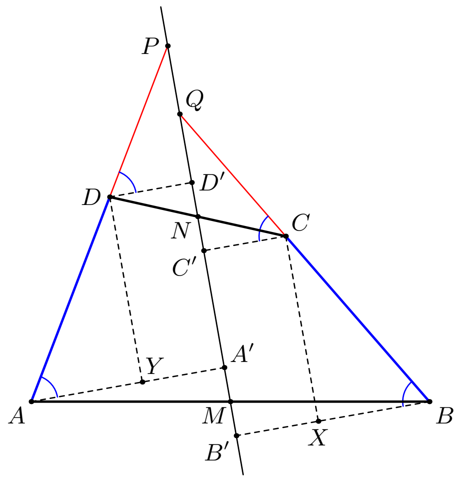
If $n = 11$, then there is a group from which we take all three numbers, i.e. their product equals m.

11. Assume that the circumcircles of triangles ADC and BEC meet at C and P. The problem is to show that the line KL makes equal angles with the lines AC and BC. Since the line joining the circumcenters of triangles ADC and BEC is perpendicular to the line CP, it suffices to show that CP is the angle-bisector of $\angle ACB$.



Since the points A, P, D, C are concyclic, we obtain $\angle EAP = \angle BDP$. Analogously, we have $\angle AEP = \angle DBP$. These two equalities together with $AE = BD$ imply that triangles APE and DPB are congruent. This means that the distance from P to AC is equal to the distance from P to BC, and thus CP is the angle-bisector of $\angle ACB$, as desired.

12.



Let A', B', C', D' be the feet of the perpendiculars from A, B, C, D, respectively, onto the line MN. Then

$$AA' = BB' \quad \text{and} \quad CC' = DD' .$$

Denote by X, Y the feet of the perpendiculars from C, D onto the lines BB', AA', respectively. We infer from the above equalities that $AY = BX$. Since also $BC = AD$, the right-angled triangles BXC and AYD are congruent. This shows that

$$\angle C'CQ = \angle B'BQ = \angle A'AP = \angle D'DP .$$

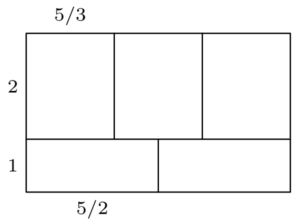
Therefore, since $CC' = DD'$, the triangles $CC'Q$ and $DD'P$ are congruent. Thus $CQ = DP$.

13. **Answer:** (a) 6 circles, (b) 5 circles.

(a) Consider the four corners and the two midpoints of the sides of length 6. The distance between any two of these six points is 3 or more, so one circle cannot cover two of these points, and at least 6 circles are needed.

On the other hand one circle will cover a 2×2 square, and it is easy to see that 6 such squares can cover the rectangle.

(b) Consider the four corners and the center of the rectangle. The minimum distance between any two of these points is the distance between the center and one of the corners, which is $\sqrt{34}/2$. This is greater than the diameter of the circle ($\sqrt{34/4} > \sqrt{32/4}$), so one circle cannot cover two of these points, and at least 5 circles are needed.



Partition the rectangle into 3 rectangles of size $5/3 \times 2$ and two rectangles of size $5/2 \times 1$ as shown above. It is easy to check that each has a diagonal of length less than $2\sqrt{2}$, so five circles can cover the five small rectangles and hence the 5×3 rectangle.

14. **Answer:** $\angle PMQ = 90°$.

Draw the parallelogram $ABCA'$, with $AA' \parallel BC$. Then M lies on BA', and $BM = \dfrac{1}{3}BA'$. So M is on the homothetic image (center B, dilation $1/3$) of the circle with center C and radius AB, which meets BC at D and E. The image meets BC at P and Q. So $\angle PMQ = 90°$.

15. Let A_1 be the intersection of a with BD, B_1 the intersection of b with AC, C_1 the intersection of c with BD and D_1 the intersection of d with AC. It follows easily by the given right angles that the following three sets each are concyclic:

- A, A_1, D, D_1, O lie on a circle w_1 with diameter AD.
- B, B_1, C, C_1, O lie on a circle w_2 with diameter BC.
- C, C_1, D, D_1 lie on a circle w_3 with diameter DC.

We see that O lies on the radical axis of w_1 and w_2. Also, Y lies on the radical axis of w_1 and w_3, and on the radical axis of w_2 and w_3, so Y is the radical center of w_1, w_2 and w_3, so it lies on the radical axis of w_1 and w_2. Analogously we prove that X lies on the radical axis of w_1 and w_2.

16. It is sufficient to show the statement for q prime. We need to prove that

$$(n+1)^p \equiv n^p \pmod{q} \Rightarrow q \equiv 1 \pmod{p}.$$

It is obvious that $(n, q) = (n+1, q) = 1$ (as n and $n+1$ cannot be divisible by q simultaneously). Hence there exists a positive integer m such that $mn \equiv 1 \pmod{q}$. In fact, m is just the multiplicative inverse of $n \pmod{q}$. Take $s = m(n+1)$. It is easy to see that

$$s^p \equiv 1 \pmod{q}.$$

Let t be the smallest positive integer which satisfies $s^t \equiv 1 \pmod{q}$ (t is the order of $s \pmod{q}$). One can easily prove that t divides p. Indeed, write $p = at + b$ where $0 \le b < t$. Then

$$1 \equiv s^p \equiv s^{at+b} \equiv \left(s^t\right)^a \cdot s^b \equiv s^b \pmod{q}.$$

By the definition of t, we must have $b = 0$. Hence t divides p. This means that $t = 1$ or $t = p$. However, $t = 1$ is easily seen to give a contradiction since then we would have

$$m(n+1) \equiv 1 \pmod{q} \text{ or } n+1 \equiv n \pmod{q}.$$

Therefore $t = p$, and p is the order of $s \pmod{q}$. By Fermat's little theorem,

$$s^{q-1} \equiv 1 \pmod{q}.$$

Since p is the order of $s \pmod{q}$, we have that p divides $q-1$, and we are done.

17. **Answer:** $a = \frac{(2m-1)^2+1}{2}$ where m is an arbitrary positive integer.

Let $y_n = 2x_n - 1$. Then $y_n = 2(2x_{n-1}x_{n-2} - x_{n-1} - x_{n-2} + 1) - 1 = 4x_{n-1}x_{n-2} - 2x_{n-1} - 2x_{n-2} + 1 = (2x_{n-1} - 1)(2x_{n-2} - 1) = y_{n-1}y_{n-2}$ when $n > 1$. Notice that $y_{n+3} = y_{n+2}y_{n+1} = y_{n+1}^2 y_n$. We see that y_{n+3} is a perfect square iff y_n is a perfect square. Hence y_{3n} is a perfect square for all $n \ge 1$ exactly when y_0 is a perfect square. Since $y_0 = 2a - 1$, the result is obtained when $a = \frac{(2m-1)^2+1}{2}$ for all positive integers m.

18. Let $x = 2^s x_1$ and $y = 2^t y_1$ where x_1 and y_1 are odd integers, contrary to assumption. Without loss of generality we can assume that $s \ge t$. We have

$$z = \frac{2^{s+t+2}}{2^t(2^{s-t}x_1 + y_1)} = \frac{2^{s+2}x_1y_1}{2^{s-t}x_1 + y_1}.$$

If $s \neq t$, then the denominator is odd and therefore z is even. So we have $s = t$ and $z = 2^{s+2}x_1y_1/(x_1+y_1)$. Let $x_1 = dx_2$, $y_1 = dy_2$ with $(x_2, y_2) = 1$. So $z = 2^{s+2}dx_2y_2/(x_2+y_2)$. As z is odd, it must be that x_2+y_2 is divisible by $2^{s+2} \geq 4$, so x_2+y_2 is divisible by 4. As x_2 and y_2 are odd integers, one of them, say x_2 is congruent to 3 modulo 4. But $(x_2, x_2+y_2) = 1$, so x_2 is a divisor of z.

19. **Answer:** Yes, it is possible.

Start with a simple Pythagorian identity:

$$3^2 + 4^2 = 5^2.$$

Multiply it with 5^2

$$3^2 \cdot 5^2 + 4^2 \cdot 5^2 = 5^2 \cdot 5^2$$

and insert the identity for the first

$$3^2 \cdot (3^2 + 4^2) + 4^2 \cdot 5^2 = 5^2 \cdot 5^2$$

which gives

$$3^2 \cdot 3^2 + 3^2 \cdot 4^2 + 4^2 \cdot 5^2 = 5^2 \cdot 5^2.$$

Multiply again with 5^2

$$3^2 \cdot 3^2 \cdot 5^2 + 3^2 \cdot 4^2 \cdot 5^2 + 4^2 \cdot 5^2 \cdot 5^2 = 5^2 \cdot 5^2 \cdot 5^2$$

and split the first

$$3^2 \cdot 3^2 \cdot (3^2 + 4^2) + 3^2 \cdot 4^2 \cdot 5^2 + 4^2 \cdot 5^2 \cdot 5^2 = 5^2 \cdot 5^2 \cdot 5^2$$

that is

$$3^2 \cdot 3^2 \cdot 3^2 + 3^2 \cdot 3^2 \cdot 4^2 + 3^2 \cdot 4^2 \cdot 5^2 + 4^2 \cdot 5^2 \cdot 5^2 = 5^2 \cdot 5^2 \cdot 5^2.$$

This (multiplying with 5^2 and splitting the first term) can be repeated as often as needed, each time increasing the number of terms by one. Clearly, each term is a square number and the terms are strictly increasing from left to right.

20. **Answer:** All numbers 2^r3^s where r and s are non-negative integers and $s \leq r \leq 2s$.

Let $m = (p_1+1)(p_2+1)\cdots(p_k+1)$. Can assume that p_k is the largest prime factor. If $p_k > 3$ then p_k cannot divide m, because if p_k divides m it is a prime factor of $p_i + 1$ for some i, but if $p_i = 2$ then $p_i + 1 < p_k$, and otherwise $p_i + 1$ is an even number with factors 2 and $\frac{1}{2}(p_i + 1)$ which are both strictly smaller than p_k. Thus the only primes that can divide n are 2 and 3, so we can write $n = 2^r3^s$. Then $m = 3^r4^s = 2^{2s}3^r$ which is divisible by n if and only if $s \leq r \leq 2s$.