Baltic Way 2003

Problems and Solutions

1. Let $\mathbb{Q}_+$ be the set of positive rational numbers.
   Find all functions $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ which for all $x \in \mathbb{Q}_+$ fulfil
   $(1) : f(\frac{1}{x}) = f(x)$
   $(2) : (1 + \frac{1}{x})f(x) = f(x+1)$

   Solution: Set $g(x) = \frac{f(x)}{f(1)}$. Function g fulfils (1), (2) and $g(1) = 1$.
   First we prove that if g exists then it is unique. We prove that g is uniquely defined on $x = \frac{p}{q}$ by induction on $\max(p, q)$. If $\max(p, q) = 1$ then $x = 1$ and $g(1) = 1$. If $p = q$ then $x = 1$ and $g(x)$ is unique. If $p \neq q$ then we can assume (according to (1)) that $p > q$. From (2) we get $g(\frac{p}{q}) = (1 + \frac{q}{p-q})g(\frac{p-q}{q})$. The induction assumption and $\max(p, q) > \max(p - q, q) \geq 1$ now give that $g(\frac{p}{q})$ is unique.
   Define the function g by $g(\frac{p}{q}) = pq$ where p and q are choosen such that $\gcd(p, q) = 1$. It is easily seen that g fulfils (1), (2) and $g(1) = 1$. All functions fulfilling (1) and (2) are therefore $f(\frac{p}{q}) = apq$, where $\gcd(p, q) = 1$ and $a \in \mathbb{Q}_+$.

2. Prove that any real solution of

   $$x^3 + px + q = 0$$

   satisfies the inequality $4qx \leq p^2$.

   Solution: Let x_0 be a root of the qubic, then $x^3 + px + q \equiv (x - x_0)(x^2 + ax + b) \equiv x^3 + (a - x_0)x^2 + (b - ax_0)x - bx_0$. So $a = x_0$, $p = b - ax_0 = b - x_0^2$, $-q = bx_0$. Hence $p^2 = b^2 - 2bx_0^2 + x_0^4$. Also $4x_0q = -4x_0^2b$. So $p^2 - 4x_0q = b^2 + 2bx_0^2 + x_0^4 = (b + x_0^2)^2 \geq 0$.

   Alternative solution: As the equation $x_0x^2 + px + q = 0$ has a root $(x = x_0)$, there must be $D \geq 0 \Leftrightarrow p^2 - 4qx_0 \geq 0$.

   (Also an equation $x^2 + px + qx_0 = 0$ having a root $x = x_0^2$ can be considered.)

3. Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

   $$(1+x)(1+y)(1+z) \geq 2\left(1 + \sqrt[3]{\frac{y}{x}} + \sqrt[3]{\frac{z}{y}} + \sqrt[3]{\frac{x}{z}}\right).$$

   Solution: Put $a = bx$, $b = cy$ and $c = az$. The given inequality then takes the form

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\sqrt[3]{\frac{b^2}{ac}}+\sqrt[3]{\frac{c^2}{ab}}+\sqrt[3]{\frac{a^2}{bc}}\right) =$$
$$= \; 2\left(1+\frac{a+b+c}{3\sqrt[3]{abc}}\right) .$$

By the A-G inequality we have

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) =$$
$$= \; \frac{a+b+c}{a}+\frac{a+b+c}{b}+\frac{a+b+c}{c}-1 \geq$$
$$\geq \; 3\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)-1 \geq 2\frac{a+b+c}{\sqrt[3]{abc}}+3-1 = 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right) ,$$

qed.

Alternative solution: Expanding the left side we obtain

$$x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 2\left(\sqrt[3]{\frac{y}{x}}+\sqrt[3]{\frac{z}{y}}+\sqrt[3]{\frac{x}{z}}\right) .$$

As $\sqrt[3]{\frac{y}{x}} \leq \frac{1}{3}\left(y+\frac{1}{x}+1\right)$ etc, it suffices to prove that

$$x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{2}{3}\left(x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+2 ,$$

which follows from $a+\frac{1}{a} \geq 2$.

4. Let a, b, c be positive real numbers. Prove that

$$\frac{2a}{a^2+bc}+\frac{2b}{b^2+ca}+\frac{2c}{c^2+ab} \leq \frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab} .$$

Solution: First we prove that

$$\frac{2a}{a^2+bc} \leq \frac{1}{2}\Big(\frac{1}{b}+\frac{1}{c}\Big),$$

which is equivalent to $0 \leq b(a-c)^2+c(a-b)^2$, and therefore holds true. Now we turn to inequality

$$\frac{1}{b}+\frac{1}{c} \leq \frac{1}{2}\Big(\frac{2a}{bc}+\frac{b}{ca}+\frac{c}{ab}\Big),$$

which is equivalent to $0 \leq (a-b)^2 + (a-c)^2$. Hence we have proved that

$$\frac{2a}{a^2+bc} \leq \frac{1}{4}\Big(\frac{2a}{bc} + \frac{b}{ca} + \frac{c}{ab}\Big).$$

Analogously we have

$$\frac{2b}{b^2+ca} \leq \frac{1}{4}\Big(\frac{2b}{ca} + \frac{c}{ab} + \frac{a}{bc}\Big),$$
$$\frac{2c}{c^2+ab} \leq \frac{1}{4}\Big(\frac{2c}{ab} + \frac{a}{bc} + \frac{b}{ca}\Big)$$

and it suffices to sum the above three inequalities.

Alternative solution: As $a^2 + bc \geq 2a\sqrt{bc}$ etc, it is sufficient to prove that

$$\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}} \leq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab},$$

which can be obtained "inserting" $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ between the left side and the right side.

5. A sequence (a_n) is defined as follows: $a_1 = \sqrt{2}$, $a_2 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \geq 2$. Prove that for every $n \geq 1$ we have

$$(1+a_1)(1+a_2)\dots(1+a_n) < (2+\sqrt{2})a_1 a_2 \dots a_n.$$

Solution: First we prove inductively that for $n \geq 1$ $a_n = 2^{2^{n-2}}$. We have $a_1 = 2^{2^{-1}}$, $a_2 = 2^{2^0}$ and

$$a_{n+1} = 2^{2^{n-2}} \cdot (2^{2^{n-3}})^2 = 2^{2^{n-2}} \cdot 2^{2^{n-2}} = 2^{2^{n-1}}.$$

Since $1 + a_1 = 1 + \sqrt{2}$, we must prove, that

$$(1+a_2)(1+a_3)\dots(1+a_n) < 2a_2 a_3 \dots a_n.$$

Right-hand side is equal to

$$2^{1+2^0+2^1+\dots+2^{n-2}} = 2^{2^{n-1}}$$

and the left-hand side

$$\begin{aligned} & (1+2^{2^0})(1+2^{2^1})\dots(1+2^{2^{n-2}}) = \\ = \ & 1 + 2^{2^0} + 2^{2^1} + 2^{2^0+2^1} + 2^{2^2} + \dots + 2^{2^0+2^1+\dots+2^{n-2}} = \\ = \ & 1 + 2 + 2^2 + 2^3 + \dots + 2^{2^{n-1}-1} = 2^{2^{n-1}} - 1. \end{aligned}$$

The proof is complete.

6. Let $n \geq 2$ and $d \geq 1$ be integers with $d \mid n$, and let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1 + x_2 + \cdots + x_n = 0$. Prove that there are at least $\binom{n-1}{d-1}$ choices of d indices $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ such that $x_{i_1} + x_{i_2} + \cdots + x_{i_d} \geq 0$.

   Solution: Put $m := n/d$ and $[n] := \{1, 2, \ldots, n\}$, and consider all partitions $[n] = A_1 \cup A_2 \cup \cdots \cup A_m$ of $[n]$ into d-element subsets A_i, $i = 1, 2, \ldots, m$. The number of such partitions is denoted by t. Clearly, there are exactly $\binom{n}{d}$ d-element subsets of $[n]$ each of which occurs in the same number of partitions. Hence, every $A \subseteq [n]$ with $|A| = d$ occurs in exactly $s := tm/\binom{n}{d}$ partitions. On the other hand, every partition contains at least one d-element set A such that $\sum_{i \in A} x_i \geq 0$. Consequently, the total number of sets with this property is at least $t/s = \binom{n}{d}/m = \frac{d}{n}\binom{n}{d} = \binom{n-1}{d-1}$.

7. Let X be a subset of $\{1, 2, 3, \ldots, 10000\}$ with the following property: if a, $b \in X$, $a \neq b$, then $a \cdot b \notin X$. What is the maximal number of elements in X?

   Solution: Answer: 9901.
   If $X = \{100, 101, 102, \ldots, 9999, 10000\}$, then for any two selected a and b, $a \neq b$, $a \cdot b \geq 100 \cdot 101 > 10000$, so $a \cdot b \notin X$. So X may have 9901 elements.

   Suppose that $x_1 < x_2 < \cdots < x_k$ are all elements of X that are less than 100. If there are none of them, no more than 9901 numbers can be in the set X. Otherwise, if $x_1 = 1$ no other number can be in the set X, so suppose $x_1 > 1$ and consider the pairs

$$
\begin{aligned}
&200 - x_1, (200 - x_1) \cdot x_1 \\
&200 - x_2, (200 - x_2) \cdot x_2 \\
&\vdots \\
&200 - x_k, (200 - x_k) \cdot x_k
\end{aligned}
$$

   Clearly $x_1 < x_2 < \cdots < x_k < 100 < 200 - x_k < 200 - x_{k-1} < \cdots < 200 - x_2 < 200 - x_1 < 200 < (200 - x_1) \cdot x_1 < (200 - x_2) \cdot x_2 < \cdots < (200 - x_k) \cdot x_k$. So all numbers in these pairs are different and greater than 100. So at most one from each pair is in the set X. Therefore, there are at least k numbers greater than 100 and $99 - k$ numbers less than 100 that are not in the set X, together at least 99 numbers out of 10000 not being in the set X.

8. There are 2003 pieces of candy on a table. Two players alternately make moves. A move consists of eating one candy or half of the candies on the table (the "lesser half" if there is an odd number of candies); at least one candy must be eaten at each move. The loser is the one who eats the last candy. Which player – the first or the second – has a winning strategy?

Solution: Answer: the second.
Let us prove inductively that for $2n$ pieces of candy the first has a winning strategy. For $n = 1$ it is obvious. Suppose it is true for $2n$ pieces, and let's consider $2n + 2$ pieces. If for $2n + 1$ pieces the second is the winner, then the first eats 1 piece and becomes the second in the game starting with $2n + 1$ pieces. So suppose that for $2n + 1$ pieces the first is the winner. His winning move for $2n + 1$ isn't eating 1 piece (accordingly to the inductive assumption). So his winning move is to eat n pieces, leaving the second with $n + 1$ pieces, when the second must lose. But the first can leave the second with $n+1$ pieces from the starting position with $2n+2$ pieces, eating $n + 1$ pieces; so $2n + 2$ is the winning position for the first.
Now if there are 2003 pieces of candy on the table, the first must eat either 1 or 1001 candies, leaving an even number of candies on the table. So the second player will be the first player in a game with even number of candies and therefore has a winning strategy.

9. It is known that n is a positive integer, $n \leq 144$. Ten questions of type "Is n smaller than a?" are allowed. Answers are given with a delay: an answer to the i-th question is given only after the $(i+1)$-st question is asked, $i = 1, 2, \dots, 9$. The answer to the 10th question is given immediately after it is asked. Find a strategy for identifying n.

   Solution: Let's denote Fibonacci numbers as $F_0 = 1$, $F_1 = 2$, $F_2 = 3$, $\dots$, $F_{10} = 144$. We will consider two types of situations: 'N' denotes that we know for sure that n is one of N consecutive integers (and we know these integers); '$N \to ?M$' denotes that we know for sure that n is one of $N + M$ consecutive integers (and we know these integers), and a question denoted by $\to ?$ is set with an answer unknown so far.
   Clearly, the initial situation is 'F_{10}'.

   Theorem. There exists a strategy which guarantees that after setting i questions and receiving answers to the first $(i - 1)$ of them $(i = 1, 2, \dots, 9)$ we get one of the following situations: 'F_{10-i}'; '$F_{9-i} \to ?F_{10-i}$'; '$F_{10-i} \to ?F_{9-i}$'.
   The proof is by a straightforward induction, the next question dividing the segment of length F_{10-i} into two segments of lengths F_{9-i} and F_{8-i}, the longest of them being situated at one or another end of the whole "segment of hypotheses".

   So after setting 9 questions we get one of the following situations (hypotetical numbers are denoted by $\circ$): '$\circ\circ$'; '$\circ \to ? \circ \circ$'; '$\circ\circ \to ?\circ$'. It is clear that with the next, 10th question, "separating" the still unseparated hypotheses, we will find n.

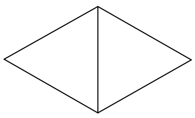
10. A *lattice point* in the plane is a point whose coordinates are both integral. The *centroid* of four points (x_i, y_i), $i = 1, 2, 3, 4$, is the point $(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4})$. Let n be the largest natural number with the following property: There are n distinct lattice points in the plane such that the centroid of any four of them is not a lattice point. Prove that $n = 12$.

    Solution: To prove $n \leq 12$, we have to show that there are 12 lattice points (x_i, y_i), $i = 1, 2, , \ldots, 12$, such that no four determine a lattice point centroid. This is guaranteed if we just choose the points such that $x_i \equiv 0 \pmod 4$ for $i = 1, \ldots, 6$, $x_i \equiv 1 \pmod 4$ for $i = 7, \ldots, 12$, $y_i \equiv 0 \pmod 4$ for $i = 1, 2, 3, 10, 11, 12$, $y_i \equiv 1 \pmod 4$ for $i = 4, \ldots, 9$.

    Now let P_i, $i = 1, 2, \ldots, 13$, be lattice points. We have to show that some four of them determine a lattice point centroid. First observe that, by the pigeonhole principle, among any five of the points we find two such that their x-coordinates as well as their y-coordinates have the same parity. Consequently, among any five of the points there are two whose midpoint is a lattice point. Iterated application of this observation implies that among the 13 points in question we find five disjoint pairs of points whose midpoint is a lattice point. Among these five midpoints we again find two, say M and M', such that their midpoint C is a lattice point. Finally, if M and M' are the midpoints of P_iP_j and P_kP_ℓ, respectively, $\{i, j, k, \ell\} \subset \{1, 2, \ldots, 13\}$, then C is the centroid of P_i, P_j, P_k, P_ℓ.

11. Is it possible to select 1000 points in a plane so that at least 6000 distances between two of them are equal?

    Solution: Yes, it is. Let's start with configuration of 4 points and 5 distances equal to d, like on picture (α):

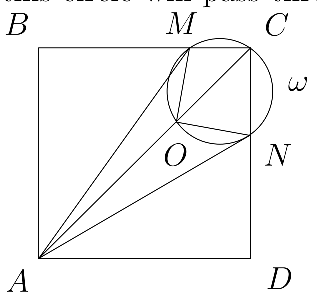
    (α)

    Now take (α) and two copies of it obtainable from (α) by parallel shifts along vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, $|\overrightarrow{a}| = |\overrightarrow{b}| = d$ and $\angle(\overrightarrow{a}, \overrightarrow{b}) = 60°$. Vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ should be chosen so that no two vertices of (α) and of two copies coincide. We get $3 \cdot 4 = 12$ points and $3 \cdot 5 + 12 = 27$ distances.
    Proceeding in the same way, we get gradually
    $3 \cdot 12 = 36$ points and $3 \cdot 27 + 36 = 117$ distances;
    $3 \cdot 36 = 108$ points and $3 \cdot 117 + 108 = 459$ distances;
    $3 \cdot 108 = 324$ points and $3 \cdot 459 + 324 = 1701$ distances;
    $3 \cdot 324 = 972$ points and $3 \cdot 1701 + 972 = 6075$ distances.

12. Let $ABCD$ be a square. Let M be an inner point on side BC and N be an

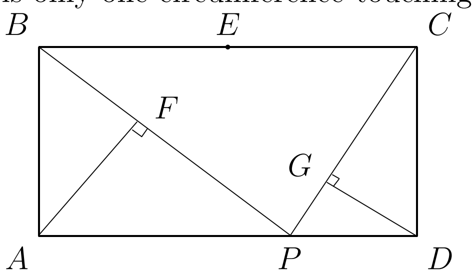
inner point on side CD with $\angle MAN = 45°$. Prove that the circumcenter of AMN lies on AC.

Solution: Draw a circle ω through M, C, N; let it intersect AC at O. We claim that O is the circumsenter of AMN.
Clearly $\angle MON = 180° - \angle MCN = 90°$. If the radius of ω is R, then $OM = 2R \sin 45° = R\sqrt{2}$; similarly $ON = R\sqrt{2}$. Therefore $OM = ON$. Draw a circle with center O and a radius $R\sqrt{2}$. As $\angle MAN = \frac{1}{2}\angle MON$, this circle will pass through A.



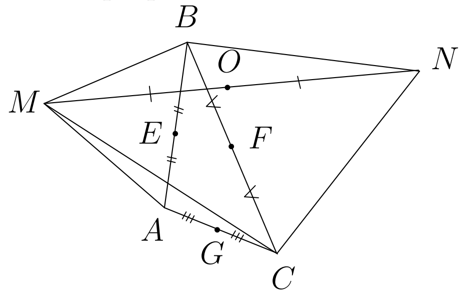
13. Let $ABCD$ be a rectangle and $BC = 2 \cdot AB$. Let E be the midpoint of BC and P an arbitrary inner point of AD. Let F and G be the feet of perpendiculars drawn correspondingly from A to BP and from D to CP. Prove that the points E, F, P, G are concyclic.

Solution: From rectangular triangle BAP we have $BP \cdot BF = AB^2 = BE^2$. Therefore the circumference through F and P touching the line BC between B and C touches it at E. Analogously, the circumference through P and G touching the line BC between B and C touches it at E. But there is only one circumference touching BC at E and passing through P.



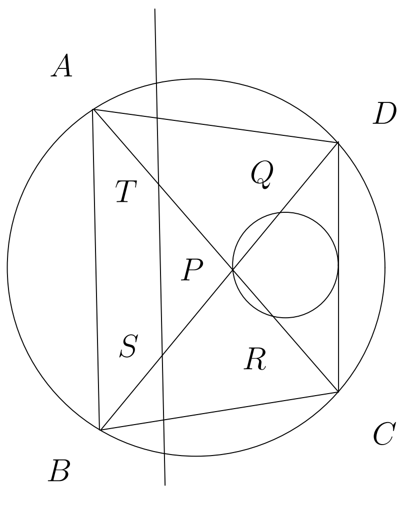
14. Let ABC be an arbitrary triangle and AMB, BNC, CKA regular triangles outward of ABC. Through the midpoint of MN a perpendicular to AC is constructed; similarly through midpoints of NK resp. KM perpendiculars to AB resp. BC are constructed. Prove that these 3 perpendiculars intersect at the same point.

Solution: Let O be the midpoint of MN, E and F – the midpoints of AB resp. BC. As $\triangle MBC$ transforms into $\triangle ABN$ when rotated for $60°$ around B we get $MC = AN$ (it is also well-known fact). Considering now the quadrangles $AMBN$ and $CMBN$ we get $OE = OF$ (from Eiler's formula $a^2+b^2+c^2+d^2 = e^2+f^2+4\cdot PQ^2$ or otherwise). As $EF \| AC$ we get from this that a perpendicular trough O passes through the circumcenter of EFG, as it is the perpendicular bisector of EF. The same holds for two other perpendiculars.



Alternative solution: Let's denote the midpoints of MN, NK, KM by B_1, C_1, A_1 respectively. Clearly $\triangle A_1B_1C_1$ is homothetic to $\triangle NKM$. The perpendiculars through M, N, K to AB, BC, CA respectively are concurrent (by radical axis, or by Steiner-Carnot theorem, or somehow else). The desired result follows now from the homothety.

15. Let P be the intersection point of the diagonals AC and BD in a cyclic quadrilateral. A circle through P touches the side CD in the midpoint M of this side and intersects the segments BD and AC in the points Q and R respectively. Let S be a point on the segment BD such that $BS = DQ$. The parallel to AB through S intersects AC at T. Prove that $AT = RC$.



Solution: With reference to the figure above we have $CR\cdot CP = DQ\cdot DP = CM^2 = DM^2 \Leftrightarrow RC = \frac{DQ\cdot DP}{CP}$. We also have $\frac{AT}{BS} = \frac{AP}{BP} = \frac{AT}{DQ} \Leftrightarrow AT = \frac{AP\cdot DQ}{BP}$. Since $ABCD$ is cyclic the result now comes from the fact that

$DP \cdot BP = AP \cdot CP$ (due to well-known theorem).

16. Find all pairs of positive integers (a, b) such that $a - b$ is a prime and ab is a perfect square.

Solution: Let p be a prime such that $a - b = p$ and let $ab = k^2$. Insert $a = b + p$ in the equation $ab = k^2$ and then do the following:

$$(b+p)b = k^2 \Leftrightarrow (b+\frac{p}{2})^2 - \frac{p^2}{4} = k^2 \Leftrightarrow (2b+p)^2 - 4k^2 = p^2 \Leftrightarrow$$
$$\Leftrightarrow (2b+p+2k)(2b+p-2k) = p^2 \,.$$

Since $2b+p+2k > 2b+p-2k$ and p is a prime, we conclude $2b+p+2k = p^2$ and $2b + p - 2k = 1$. By adding these equations we get $2b + p = \frac{p^2+1}{2}$ and then $b = (\frac{p-1}{2})^2$. $a = b + p = (\frac{p+1}{2})^2$. By checking we conclude that all the solutions are $(a, b) = ((\frac{p+1}{2})^2, (\frac{p-1}{2})^2)$ with p a prime greater than 2.

Alternative solution: Let p be a prime such that $a - b = p$ and let $ab = k^2$. We have $(b + p)b = k^2$; $\gcd(b, b + p) = \gcd(b, p)$ is equal either to 1 or p.

(1) $\gcd(b, b+p) = p$. Let $b = b_1p$. Then $p^2b_1(b_1 + 1) = k^2$, $b_1(b_1 + 1) = m^2$, this equation has no solutions.

(2) $\gcd(b, b + p) = 1$. Then

$$\begin{cases} b = u^2 \\ b + p = v^2 \end{cases} \Rightarrow p = u^2 - v^2 = (u - v)(u + v) \Rightarrow$$
$$\Rightarrow u - v = 1, u + v = p \Rightarrow$$
$$\Rightarrow a = \left(\frac{p+1}{2}\right)^2, b = \left(\frac{p-1}{2}\right)^2 ;$$

where p must be an odd prime.

17. All the positive divisors of a positive integer n are stored into an array in increasing order. Mary has to write a program which decides for an arbitrarily chosen divisor $d > 1$ whether it is a prime. Let n have k divisors not greater than d. Mary claims that it suffices to check divisibility of d by the first $\lceil k/2 \rceil$ divisors of n: if a divisor of d greater than 1 is found among them, then d is composite, otherwise d is prime. Is Mary right?

Solution: Yes, Mary is right.

Let $d > 1$ be a divisor of n.

Suppose Mary's program outputs "composite" for d. That means it has found a divisor of d greater than 1. Since $d > 1$, the array contains at least 2 divisors of d: 1 and d. Thus Mary's program does not check divisibility of

d by d (the first half gets complete before reaching d) which means that the divisor found lays strictly between 1 and d. Hence d is composite indeed.

Suppose now d being composite. Let p be its smallest prime divisor; then $\frac{d}{p} \geq p$ or, equivalently, $d \geq p^2$. As p is a divisor of n, it occurs in the array. Let $a_1, \ldots, a_k$ all divisors of n smaller than p. Then $pa_1, \ldots, pa_k$ are less than p^2 and hence less than d. As $a_1, \ldots, a_k$ are all relatively prime with p, all the numbers $pa_1, \ldots, pa_k$ divide n. The numbers $a_1, \ldots, a_k, pa_1, \ldots, pa_k$ are pairwise different by construction. Thus there are at least $2k+1$ divisors of n not greater than d. So Mary's program checks divisibility of d by at least $k+1$ smallest divisors of n, among which it finds p, and outputs "composite".

18. Every integer is colored with exactly one of the colors BLUE, GREEN, RED, YELLOW. Can this be done in such a way that if a, b, c, d are not all 0 and have the same color, then $3a - 2b \neq 2c - 3d$?

    Solution: The answer is yes. A coloring with the required property can be defined as follows. For an integer k let k^* be the integer uniquely defined by $k = 5^m \cdot k^*$, where m is a nonnegative integer and $5 \not| k^*$. Two integers k_1, k_2 receive the same color if and only if $k_1^* \equiv k_2^* \pmod 5$.

    Assume that $3a - 2b = 2c - 3d$, i.e. $3a - 2b - 2c + 3d = 0$. Dividing both sides by the largest power of 5 which simultaneously divides a, b, c, d, we obtain

    $$3 \cdot 5^A \cdot a^* - 2 \cdot 5^B \cdot b^* - 2 \cdot 5^C \cdot c^* + 3 \cdot 5^D \cdot d^* = 0,$$

    where A, B, C, D are nonnegative integers at least one of which is equal to 0. The above equality implies

    $$3(5^A \cdot a^* + 5^B \cdot b^* + 5^C \cdot c^* + 5^D \cdot d^*) \equiv 0 \pmod 5.$$

    If a, b, c, d all had the same color, then $a^* \equiv b^* \equiv c^* \equiv d^* \not\equiv 0 \pmod 5$ would hold. This implies

    $$5^A + 5^B + 5^C + 5^D \equiv 0 \pmod 5$$

    which is impossible since at least one of the numbers A, B, C, D is equal to 0.

19. Let a and b be positive integers. Prove that if $a^3 + b^3$ is the square of an integer, then $a + b$ is not a product of two different prime numbers.

    Solution: Suppose $a + b = pq$, where $p \neq q$ are two prime numbers. We may assume that $p \neq 3$. Since $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ is a square, the number $a^2 - ab + b^2 = (a+b)^2 - 3ab$ must be divisible by p and q, and hence $3ab$ must be divisible by p and q. But $p \neq 3$, so $p|a$ or $p|b$; but $p|a+b$, so $p|a$ and $p|b$: $a = pk$, $b = p\ell$ for some integers k, ℓ. Notice that $q = 3$, since otherwise, repeating the above argument, we would have $q|a$, $q|b$ and $a + b > pq$). So we have

$$3p = a + b = p(k + \ell)$$

and we conclude that $a = p$, $b = 2p$ or $a = 2p$, $b = p$. Then $a^3 + b^3 = 9p^3$ is obviously not a square, a contradiction.

20. Let n be a positive integer such that the sum of all positive divisors of n (except n) plus the number of these divisors is equal to n. Prove that $n = 2m^2$ for some integer m.

    Solution: Let $t_1, t_2, \ldots, t_s$ be all potitive odd divisors of n, 2^k be the maximal power of 2 that divides n. Then the full list of divisors of n is the following:

    $$t_1, \ \ldots, \ t_s, \ 2t_1, \ \ldots, \ 2t_s, \ \ldots 2^k t_1, \ \ldots, \ 2^k t_s \, .$$

    Hence,

    $$2n = (2^{k+1} - 1)(t_1 + t_2 + \ldots + t_s) + (k+1)s - 1 \, .$$

    The right hand side can be even only if both k and s are odd. In this case the number $n/2^k$ has odd number of divisors and therefore it is equal to a perfect square.