Baltic Way 2001

Hamburg, November 4, 2001

Problems

1. A set of 8 problems was prepared for an examination. Each student was given 3 of them. No two students received more than one common problem. What is the largest possible number of students?

2. Let $n \geqslant 2$ be a positive integer. Find whether there exist n pairwise nonintersecting nonempty subsets of $\{1, 2, 3, \ldots\}$ such that each positive integer can be expressed in a unique way as a sum of at most n integers, all from different subsets.

3. The numbers 1, 2, ..., 49 are placed in a 7×7 array, and the sum of the numbers in each row and in each column is computed. Some of these 14 sums are odd while others are even. Let A denote the sum of all the odd sums and B the sum of all even sums. Is it possible that the numbers were placed in the array in such a way that $A = B$?

4. Let p and q be two different primes. Prove that

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \left\lfloor \frac{3p}{q} \right\rfloor + \ldots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \frac{1}{2}(p-1)(q-1) .$$

   (Here $\lfloor x \rfloor$ denotes the largest integer not greater than x.)

5. Let 2001 given points on a circle be colored either red or green. In one step all points are recolored simultaneously in the following way: If both direct neighbors of a point P have the same color as P, then the color of P remains unchanged, otherwise P obtains the other color. Starting with the first coloring F_1, we obtain the colorings $F_2, F_3, \ldots$ after several recoloring steps. Prove that there is a number $n_0 \leqslant 1000$ such that $F_{n_0} = F_{n_0+2}$. Is the assertion also true if 1000 is replaced by 999?

6. The points A, B, C, D, E lie on the circle c in this order and satisfy $AB \parallel EC$ and $AC \parallel ED$. The line tangent to the circle c at E meets the line AB at P. The lines BD and EC meet at Q. Prove that $|AC| = |PQ|$.

7. Given a parallelogram $ABCD$. A circle passing through A meets the line segments AB, AC and AD at inner points M, K, N, respectively. Prove that

$$|AB| \cdot |AM| + |AD| \cdot |AN| = |AK| \cdot |AC| .$$

8. Let $ABCD$ be a convex quadrilateral, and let N be the midpoint of BC. Suppose further that $\angle AND = 135^\circ$. Prove that

$$|AB| + |CD| + \frac{1}{\sqrt{2}} \cdot |BC| \geqslant |AD| .$$

9. Given a rhombus $ABCD$, find the locus of the points P lying inside the rhombus and satisfying $\angle APD + \angle BPC = 180^\circ$.

10. In a triangle ABC, the bisector of $\angle BAC$ meets the side BC at the point D. Knowing that $|BD| \cdot |CD| = |AD|^2$ and $\angle ADB = 45^\circ$, determine the angles of triangle ABC.

11. The real-valued function f is defined for all positive integers. For any integers $a > 1$, $b > 1$ with $d = \gcd(a, b)$, we have

$$f(ab) = f(d) \cdot \left(f\left(\frac{a}{d}\right) + f\left(\frac{b}{d}\right) \right) ,$$

Determine all possible values of $f(2001)$.

12. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $\sum_{i=1}^{n} a_i^3 = 3$ and $\sum_{i=1}^{n} a_i^5 = 5$. Prove that $\sum_{i=1}^{n} a_i > \frac{3}{2}$.

13. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_0 = 1$ and $a_n = a_{\lfloor 7n/9 \rfloor} + a_{\lfloor n/9 \rfloor}$ for $n = 1, 2, \ldots$. Prove that there exists a positive integer k with $a_k < \frac{k}{2001!}$.

(Here $\lfloor x \rfloor$ denotes the largest integer not greater than x.)

14. There are $2n$ cards. On each card some real number x, $1 \leqslant x \leqslant 2$, is written (there can be different numbers on different cards). Prove that

the cards can be divided into two heaps with sums s_1 and s_2 so that $\frac{n}{n+1} \leqslant \frac{s_1}{s_2} \leqslant 1$.

15. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $i \cdot a_i^2 \geqslant (i+1) \cdot a_{i-1}a_{i+1}$ for $i = 1, 2, \ldots$. Furthermore, let x and y be positive reals, and let $b_i = xa_i + ya_{i-1}$ for $i = 1, 2, \ldots$. Prove that the inequality $i \cdot b_i^2 > (i+1) \cdot b_{i-1}b_{i+1}$ holds for all integers $i \geqslant 2$.

16. Let f be a real-valued function defined on the positive integers satisfying the following condition: For all $n > 1$ there exists a prime divisor p of n such that

$$f(n) = f\left(\frac{n}{p}\right) - f(p) .$$

Given that $f(2001) = 1$, what is the value of $f(2002)$?

17. Let n be a positive integer. Prove that at least $2^{n-1} + n$ numbers can be chosen from the set $\{1, 2, 3, \ldots, 2^n\}$ such that for any two different chosen numbers x and y, $x + y$ is not a divisor of $x \cdot y$.

18. Let a be an odd integer. Prove that $a^{2^n} + 2^{2^n}$ and $a^{2^m} + 2^{2^m}$ are relatively prime for all positive integers n and m with $n \neq m$.

19. What is the smallest positive odd integer having the same number of positive divisors as 360?

20. From a sequence of integers (a, b, c, d) each of the sequences

$$(c, d, a, b),\ (b, a, d, c),\ (a{+}nc, b{+}nd, c, d),\ (a{+}nb, b, c{+}nd, d) ,$$

for arbitrary integer n can be obtained by one step. Is it possible to obtain (3, 4, 5, 7) from (1, 2, 3, 4) through a sequence of such steps?

Solutions

1. *Answer:* 8.

   Denote the problems by A, B, C, D, E, F, G, H, then 8 possible problem sets are ABC, ADE, AFG, BDG, BFH, CDH, CEF, EGH. Hence, there could be 8 students.

Suppose that some problem (e.g., A) was given to 4 students. Then each of these 4 students should receive 2 different "supplementary" problems, and there should be at least 9 problems — a contradiction. Therefore each problem was given to at most 3 students, and there were at most $8 \cdot 3 = 24$ "awardings" of problems. As each student was "awarded" 3 problems, there were at most 8 students.

2. *Answer:* yes.

   Let A_1 be the set of positive integers whose only non-zero digits may be the 1-st, the $(n+1)$-st, the $(2n+1)$-st etc. from the end; A_2 be the set of positive integers whose only non-zero digits may be the 2-nd, the $(n+2)$-nd, the $(2n+2)$-nd etc. from the end, and so on. The sets A_1, A_2, ..., A_n have the required property.

   Remark. This problem is quite similar to problem 18 from Baltic Way 1997.

3. *Answer:* no.

   If this were possible, then $2 \cdot (1 + \ldots + 49) = A + B = 2B$. But B is even since it is the sum of even numbers, whereas $1 + \ldots + 49 = 25 \cdot 49$ is odd. This is a contradiction.

4. The line $y = \frac{p}{q}x$ contains the diagonal of the rectangle with vertices $(0,0)$, $(q,0)$, (q,p) and $(0,p)$ and passes through no points with integer coordinates in the interior of that rectangle. For $k = 1, 2, \ldots, q-1$ the summand $\left\lfloor \frac{kp}{q} \right\rfloor$ counts the number of interior points of the rectangle lying below the diagonal $y = \frac{p}{q}x$ and having x-coordinate equal to k. Therefore the sum in consideration counts all interior points with integer coordinates below the diagonal, which is exactly half the number of all points with integer coordinates in the interior of the rectangle, i.e. $\frac{1}{2} \cdot (p-1)(q-1)$.

   Remark. The integers p and q need not be primes: in the solution we only used the fact that they are coprime.

5. *Answer:* no.

   Let the points be denoted by 1, 2, ..., 2001 such that i, j are neighbors if $|i-j| = 1$ or $\{i, j\} = \{1, 2001\}$. We say that k points form a *monochromatic segment of length k* if the points are consecutive on the circle and if they all have the same color. For a coloring F let $d(F)$ be the maximum length of a monochromatic segment. Note that $d(F_n) > 1$ for all n since 2001 is odd. If $d(F_1) = 2001$ then all points have the same color, hence $F_1 = F_2 = F_3 = \dots$ and we can choose $n_0 = 1$. Thus, let $1 < d(F_1) < 2001$. Below we shall prove the following implications:

$$\text{If } 3 < d(F_n) < 2001, \text{ then } d(F_{n+1}) = d(F_n) - 2\ ; \tag{1}$$

$$\text{If } d(F_n) = 3, \text{ then } d(F_{n+1}) = 2\ ; \tag{2}$$

$$\text{If } d(F_n) = 2, \text{ then } d(F_{n+1}) = d(F_n) \text{ and } F_{n+2} = F_n\ ; \tag{3}$$

From (1) and (2) it follows that $d(F_{1000}) \leqslant 2$, hence by (3) we have $F_{1000} = F_{1002}$. Moreover, if F_1 is the coloring where 1 is colored red and all other points are colored green, then $d(F_1) = 2000$ and thus $d(F_1) > d(F_2) > \dots > d(F_{1000}) = 2$ which shows that, for all $n < 1000, F_n \neq F_{n+2}$ and thus 1000 cannot be replaced by 999.

It remains to prove (1)–(3). Let $(i+1, \dots, i+k)$ be a longest monochromatic segment for F_n (considering the labels of the points modulo 2001). Then $(i+2, \dots, i+k-1)$ is a monochromatic segment for F_{n+1} and thus $d(F_{n+1}) \geqslant d(F_n) - 2$. Moreover, if $(i+1, \dots, i+k)$ is a longest monochromatic segment for F_{n+1} where $k \geqslant 3$, then $(i, \dots, i+k+1)$ is a monochromatic segment for F_n. From this and $F_{n+1} > 1$ the implications (1) and (2) clearly follow. For proof of (3) note that if $d(F_n) \leqslant 2$ then F_{n+1} is obtained from F_n by changing the colour of all points.

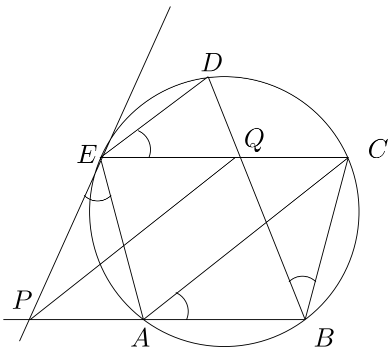

Figure 1

6. The arcs BC and AE are of equal length (see Figure 1). Also, since

$AB \parallel EC$ and $ED \parallel AC$, we have $\angle CAB = \angle DEC$ and the arcs DC and BC are of equal length. Since PE is tangent to c and $|AE| = |DC|$, then $\angle PEA = \angle DBC = \angle QBC$. As $ABCD$ is inscribed in c, we have $\angle QCB = 180° - \angle EAB = \angle PAE$. Also, $ABCD$ is an isosceles trapezium, whence $|AE| = |BC|$. So the triangles APE and CQB are congruent, and $|QC| = |PA|$. Now $PACQ$ is a quadrilateral with a pair of opposite sides equal and parallel. So $PACQ$ is a parallelogram, and $|PQ| = |AC|$.

7. Let X be the point on segment AC such that $\angle ADX = \angle AKN$, then

$$\angle AXD = \angle ANK = 180° - \angle AMK$$

(see Figure 2). Triangles NAK and XAD are similar, having two pairs of equal angles, hence $|AX| = \dfrac{|AN| \cdot |AD|}{|AK|}$. Since triangles MAK and XCD are also similar, we have $|CX| = \dfrac{|AM| \cdot |CD|}{|AK|} = \dfrac{|AM| \cdot |AB|}{|AK|}$ and

$$|AM| \cdot |AB| + |AN| \cdot |AD| = (|AX| + |CX|) \cdot |AK| = |AC| \cdot |AK|.$$

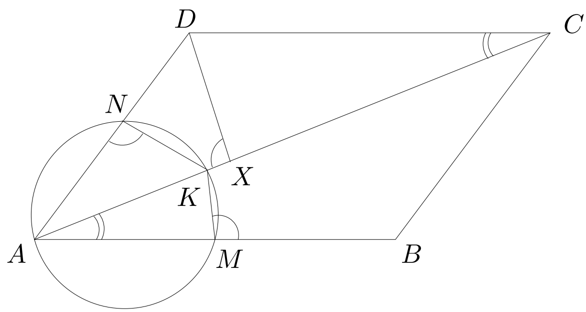

Figure 2

8. Let X be the point symmetric to B with respect to AN, and let Y be the point symmetric to C with respect to DN (see Figure 3). Then

$$\angle XNY = 180° - 2 \cdot (180° - 135°) = 90°$$

and $|NX| = |NY| = \dfrac{|BC|}{2}$. Therefore, $|XY| = \dfrac{|BC|}{\sqrt{2}}$. Moreover, we have

$|AX| = |AB|$ and $|DY| = |DC|$. Consequently,

$$|AD| \leqslant |AX| + |XY| + |YD| = |AB| + \frac{|BC|}{\sqrt{2}} + |DC| .$$

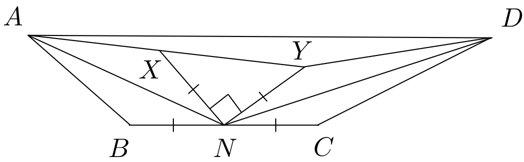

Figure 3

9. *Answer:* the locus of the points P is the union of the diagonals AC and BD.

   Let Q be a point such that $PQCD$ is a parallelogram (see Figure 4). Then $ABQP$ is also a parallelogram. From the equality $\angle APD + \angle BPC = 180^\circ$ it follows that $\angle BQC + \angle BPC = 180^\circ$, so the points B, Q, C, P lie on a common circle. Therefore, $\angle PBC = \angle PQC = \angle PDC$, and since $|BC| = |CD|$, we obtain that $\angle CPB = \angle CPD$ or $\angle CPB + \angle CPD = 180^\circ$. Hence, the point P lies on the segment AC or on the segment BD.

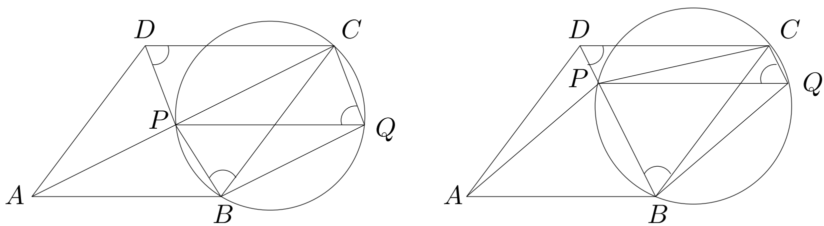

Figure 4

   Conversely, any point P lying on the diagonal AC satisfies the equation $\angle BPC = \angle DPC$. Therefore, $\angle APD + \angle BPC = 180^\circ$. Analogously, we show that the last equation holds if the point P lies on the diagonal BD.

10. *Answer:* $\angle BAC = 60^\circ$, $\angle ABC = 105^\circ$ and $\angle ACB = 15^\circ$.

    Suppose the line AD meets the circumcircle of triangle ABC at A and E (see Figure 5). Let M be the midpoint of BC and O the circumcentre of triangle ABC. Since the arcs BE and EC are equal, then the points O, M, E are collinear and OE is perpendicular to BC. From the equality

$\angle CDE = \angle ADB = 45^\circ$ it follows that $\angle AEO = 45^\circ$. Since $|AO| = |EO|$, we have $\angle AOE = 90^\circ$ and $AO \parallel DM$.

From the equality $|BD| \cdot |CD| = |AD|^2$ we obtain $|AD| = |DE|$, which implies that $|OM| = |ME|$. Therefore $|BO| = |BE|$ and also $|BO| = |EO|$. Hence the triangle BOE is equilateral. This gives $\angle BAE = 30^\circ$, so $\angle BAC = 60^\circ$. Summing up the angles of the triangle ABD we obtain $\angle ABC = 105^\circ$ and from this $\angle ACB = 15^\circ$.

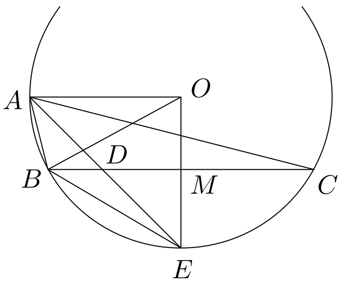

Figure 5

11. *Answer:* 0 and $\frac{1}{2}$.

Obviously the constant functions $f(n) = 0$ and $f(n) = \frac{1}{2}$ provide solutions. We show that there are no other solutions. Assume $f(2001) \neq 0$. Since $2001 = 3 \cdot 667$ and $\gcd(3, 667) = 1$, then

$$f(2001) = f(1) \cdot (f(3) + f(667)),$$

and $f(1) \neq 0$. Since $\gcd(2001, 2001) = 2001$ then

$$f(2001^2) = f(2001)(2 \cdot f(1)) \neq 0.$$

Also $\gcd(2001, 2001^3) = 2001$, so

$$f(2001^4) = f(2001) \cdot (f(1) + f(2001^2)) = f(1)f(2001)(1 + 2f(2001)).$$

On the other hand, $\gcd(2001^2, 2001^2) = 2001^2$ and

$$f(2001^4) = f(2001^2) \cdot (f(1) + f(1)) = 2f(1)f(2001^2) = 4f(1)^2 f(2001).$$

So $4f(1) = 1 + 2f(2001)$ and $f(2001) = 2f(1) - \frac{1}{2}$. Exactly the same argument starting from $f(2001^2) \neq 0$ instead of $f(2001)$ shows that $f(2001^2) = 2f(1) - \dfrac{1}{2}$. So

$$2f(1) - \frac{1}{2} = 2f(1)\Big(2f(1) - \frac{1}{2}\Big) .$$

Since $2f(1) - \dfrac{1}{2} = f(2001) \neq 0$, we have $f(1) = \dfrac{1}{2}$, which implies $f(2001) = 2f(1) - \dfrac{1}{2} = \dfrac{1}{2}$.

12. By Hölder's inequality,

$$\sum_{i=1}^{n} a^3 = \sum_{i=1}^{n}(a_i \cdot a_i^2) \leqslant \left(\sum_{i=1}^{n} a_i^{5/3}\right)^{3/5} \cdot \left(\sum_{i=1}^{n}(a_i^2)^{5/2}\right)^{2/5} .$$

We will show that

$$\sum_{i=1}^{n} a_i^{5/3} \leqslant \left(\sum_{i=1}^{n} a_i\right)^{5/3} . \tag{4}$$

Let $S = \sum_{i=1}^{n} a_i$, then (4) is equivalent to

$$\sum_{i=1}^{n}\left(\frac{a_i}{S}\right)^{5/3} \leqslant 1 = \sum_{i=1}^{n} \frac{a_i}{S} ,$$

which holds since $0 < \dfrac{a_i}{S} \leqslant 1$ and $\dfrac{5}{3} > 1$ yield $\left(\dfrac{a_i}{S}\right)^{5/3} \leqslant \dfrac{a_i}{S}$. So,

$$\sum_{i=1}^{n} a_i^3 \leqslant \left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} a_i^5\right)^{2/5} ,$$

which gives $\sum_{i=1}^{n} a_i \geqslant \dfrac{3}{5^{2/5}} > \dfrac{3}{2}$, since $2^5 > 5^2$ and hence $2 > 5^{2/5}$.

13. Consider the equation

$$\left(\frac{7}{9}\right)^x + \left(\frac{1}{9}\right)^x = 1 .$$

It has a root $\frac{1}{2} < \alpha < 1$, because $\sqrt{\frac{7}{9}} + \sqrt{\frac{1}{9}} = \frac{\sqrt{7}+1}{3} > 1$ and $\frac{7}{9} + \frac{1}{9} < 1$.

We will prove that $a_n \leqslant M \cdot n^\alpha$ for some $M > 0$ — since $\frac{n^\alpha}{n}$ will be arbitrarily small for large enough n, the claim follows from this immediately. We choose M so that the inequality $a_n \leqslant M \cdot n^\alpha$ holds for $1 \leqslant n \leqslant 8$; since for $n \geqslant 9$ we have $1 < [7n/9] < n$ and $1 \leqslant [n/9] < n$, it follows by induction that

$$
\begin{aligned}
a_n &= a_{[7n/9]} + a_{[n/9]} \leqslant M \cdot \left[\frac{7n}{9}\right]^\alpha + M \cdot \left[\frac{n}{9}\right]^\alpha \leqslant \\
&\leqslant M \cdot \left(\frac{7n}{9}\right)^\alpha + M \cdot \left(\frac{n}{9}\right)^\alpha = M \cdot n^\alpha \cdot \left(\left(\frac{7}{9}\right)^\alpha + \left(\frac{1}{9}\right)^\alpha\right) = M \cdot n^\alpha .
\end{aligned}
$$

14. Let the numbers be $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{2n-1} \leqslant x_{2n}$. We will show that the choice $s_1 = x_1 + x_3 + x_5 + \cdots + x_{2n-1}$ and $s_2 = x_2 + x_4 + \cdots + x_{2n}$ solves the problem. Indeed, the inequality $\frac{s_1}{s_2} \leqslant 1$ is obvious and we have

$$
\begin{aligned}
\frac{s_1}{s_2} &= \frac{x_1 + x_3 + x_5 + \ldots + x_{2n-1}}{x_2 + x_4 + x_6 + \ldots + x_{2n}} = \frac{(x_3 + x_5 + \ldots + x_{2n-1}) + x_1}{(x_2 + x_4 + \ldots + x_{2n-2}) + x_{2n}} \geqslant \\
&\geqslant \frac{(x_3 + x_5 + \ldots + x_{2n-1}) + 1}{(x_2 + x_4 + \ldots + x_{2n-2}) + 2} \geqslant \frac{(x_2 + x_4 + \ldots + x_{2n-2}) + 1}{(x_2 + x_4 + \ldots + x_{2n-2}) + 2} = \\
&= 1 - \frac{1}{(x_2 + x_4 + \ldots + x_{2n-2}) + 2} \geqslant 1 - \frac{1}{(n-1) + 2} = \frac{n}{n+1} .
\end{aligned}
$$

15. Let $i \geqslant 2$. We are given the inequalities

$$(i-1) \cdot a_{i-1}^2 \geqslant i \cdot a_i a_{i-2} \tag{5}$$

and

$$i \cdot a_i^2 \geqslant (i+1) \cdot a_{i+1} a_{i-1} . \tag{6}$$

Multiplying both sides of (6) by x^2, we obtain

$$i \cdot x^2 \cdot a_i^2 \geqslant (i+1) \cdot x^2 \cdot a_{i+1} a_{i-1} . \tag{7}$$

By (5),

$$\frac{a_{i-1}^2}{a_i a_{i-2}} \geqslant \frac{i}{i-1} = 1 + \frac{1}{i-1} > 1 + \frac{1}{i} = \frac{i+1}{i} ,$$

which implies

$$i \cdot y^2 \cdot a_{i-1}^2 > (i+1) \cdot y^2 \cdot a_i a_{i-2} . \qquad (8)$$

Multiplying (5) and (6), and dividing both sides of the resulting inequality by $ia_i a_{i-1}$, we get

$$(i-1) \cdot a_i a_{i-1} \geqslant (i+1) \cdot a_{i+1} a_{i-2} .$$

Adding $(i+1)a_i a_{i-1}$ to both sides of the last inequality and multiplying both sides of the resulting inequality by xy gives

$$i \cdot 2xy \cdot a_i a_{i-1} \geqslant (i+1) \cdot xy \cdot (a_{i+1} a_{i-2} + a_i a_{i-1}) . \qquad (9)$$

Finally, adding up (7), (8) and (9) results in

$$i \cdot (xa_i + ya_{i-1})^2 > (i+1) \cdot (xa_{i+1} + ya_i)(xa_{i-1} + ya_{i-2}) ,$$

which is equivalent to the claim.

16. *Answer:* 2.

For any prime p we have $f(p) = f(1) - f(p)$ and thus $f(p) = \dfrac{f(1)}{2}$. If n is a product of two primes p and q, then $f(n) = f(p) - f(q)$ or $f(n) = f(q) - f(p)$, so $f(n) = 0$. By the same reasoning we find that if n is a product of three primes, then there is a prime p such that

$$f(n) = f\left(\frac{n}{p}\right) - f(p) = -f(p) = -\frac{f(1)}{2} .$$

By simple induction we can show that if n is the product of k primes, then $f(n) = (2-k) \cdot \dfrac{f(1)}{2}$. In particular, $f(2001) = f(3 \cdot 23 \cdot 29) = 1$ so $f(1) = -2$. Therefore, $f(2002) = f(2 \cdot 7 \cdot 11 \cdot 13) = -f(1) = 2$.

17. We choose the numbers 1, 3, 5, ..., $2^n - 1$ and 2, 4, 8, 16, ..., 2^n, i.e. all odd numbers and all powers of 2. Consider the three possible cases.

(1) If $x = 2a-1$ and $y = 2b-1$, then $x+y = (2a-1)+(2b-1) = 2(a+b-1)$ is even and does not divide $xy = (2a-1)(2b-1)$ which is odd.

(2) If $x = 2^k$ and $y = 2^m$ where $k < m$, then $x + y = 2^k(2^{m-k} + 1)$ has an odd divisor greater than 1 and hence does not divide $xy = 2^{a+b}$.

(3) If $x = 2^k$ and $y = 2b - 1$, then $x + y = 2^k + (2b - 1) > (2b - 1)$ is odd and hence does not divide $xy = 2^k(2b - 1)$ which has $2b - 1$ as its largest odd divisor.

18. Rewriting $a^{2^n} + 2^{2^n} = a^{2^n} - 2^{2^n} + 2 \cdot 2^{2^n}$ and making repeated use of the identity

$$a^{2^n} - 2^{2^n} = (a^{2^{n-1}} - 2^{2^{n-1}}) \cdot (a^{2^{n-1}} + 2^{2^{n-1}})$$

we get

$$a^{2^n} + 2^{2^n} = (a^{2^{n-1}} + 2^{2^{n-1}}) \cdot (a^{2^{n-2}} + 2^{2^{n-2}}) \cdot \ldots \cdot (a^{2^m} + 2^{2^m}) \cdot \ldots$$
$$\ldots \cdot (a^2 + 2^2) \cdot (a + 2) \cdot (a - 2) + 2 \cdot 2^{2^n} .$$

For $n > m$, assume that $a^{2^n} + 2^{2^n}$ and $a^{2^m} + 2^{2^m}$ have a common divisor $d > 1$. Then an odd integer d divides $2 \cdot 2^{2^n}$, a contradiction.

19. *Answer:* 31185.

An integer with the prime factorization $p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_k^{r_k}$ (where p_1, p_2, ..., p_k are distinct primes) has precisely $(r_1 + 1) \cdot (r_2 + 1) \cdot \ldots \cdot (r_k + 1)$ distinct positive divisors. Since $360 = 2^3 \cdot 3^2 \cdot 5$, it follows that 360 has $4 \cdot 3 \cdot 2 = 24$ positive divisors. Since $24 = 3 \cdot 2 \cdot 2 \cdot 2$, it is easy to check that the smallest odd number with 24 positive divisors is $3^2 \cdot 5 \cdot 7 \cdot 11 = 31185$.

20. *Answer:* no.

Under all transformations $(a, b, c, d) \to (a', b', c', d')$ allowed in the problem we have $|ad - bc| = |a'd' - b'c'|$, but $|1 \cdot 4 - 2 \cdot 3| = 2 \neq 1 = |3 \cdot 7 - 4 \cdot 5|$.

Remark. The transformations allowed in the problem are in fact the elementary transformations of the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and the invariant $|ad - bc|$ is the absolute value of the determinant which is preserved under these transformations.